602953

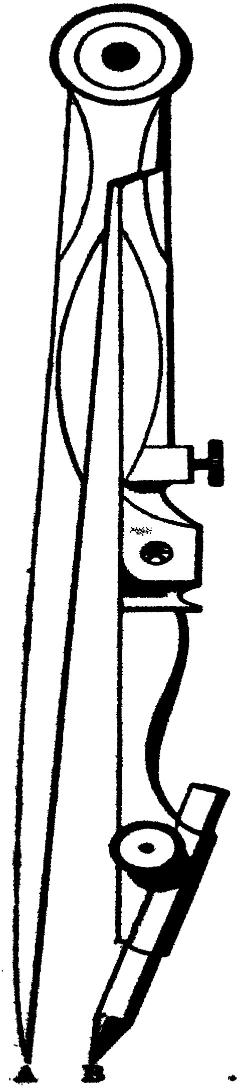


Fig. 1

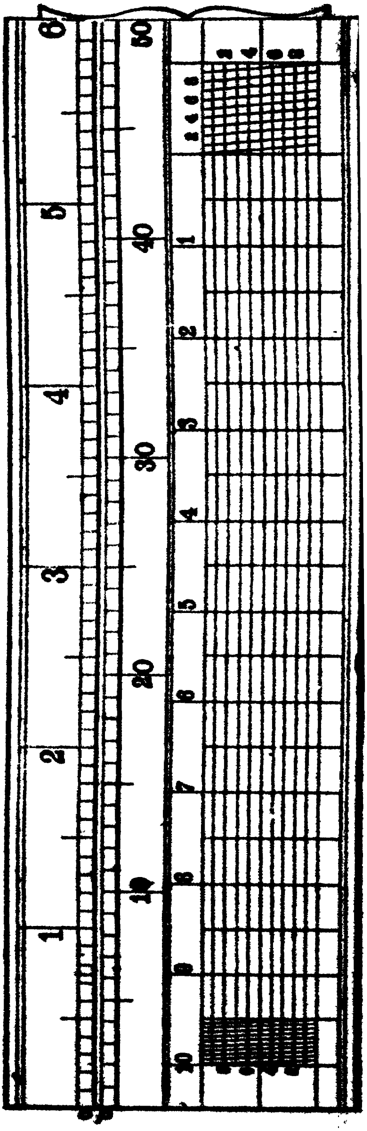


Fig. 2.

OLNEY'S MATHEMATICAL SERIES.

A TREATISE

ON

SPECIAL OR ELEMENTARY

GEOMETRY.

UNIVERSITY EDITION.

INCLUDING AN ELEMENTARY, AND ALSO, IN PART III., A HIGHER COURSE, I
PLANE, SOLID, AND SPHERICAL GEOMETRY; AND PLANE AND SPHERICAL
TRIGONOMETRY, WITH THE NECESSARY TABLES.

BY

EDWARD OLNEY,

PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF MICHIGAN.

NEW YORK:
SHELDON & COMPANY,
No. 8 MURRAY STREET.
1876.

PREFACE.

THIS treatise on the *Special* or *Elementary Geometry* consists of four parts.

PART I. is designed as an introduction. In it the student is made familiar with the geometrical concepts, and with the fundamental definitions and facts of the science. The definitions here given, are given once for all. It is thought that the pupil can obtain his first conception of a geometrical fact, as well, at least, from a correct scientific statement of it, as from some crude, colloquial form, the language of which he will be obliged to replace by better, after the former shall have become so firmly fixed in his mind, as not to be easily eradicated. No attempt at demonstration is made in this part although most of the fundamental facts of Elementary Plane Geometry are here presented, and amply and familiarly illustrated. This course has been taken in obedience to the canon of the teacher's art which prescribes "facts before theories." Moreover, such has been the historic order of development of this, and most other sciences; viz., the *facts* have been known, or conjectured, long before men have been able to give any logical account of them. And does not this indicate what *may* be the natural order in which the individual mind will receive science? When the student has become familiar with the things (concepts) about which his mind is to be occupied, and knows some of the more important of their properties and relations he is better prepared to reason upon them.

PART II. contains all the essential propositions in Plane, Solid, and Spherical Geometry, which are found in our common text-books, with their demonstrations. The subject of triedrals and the doctrine of the sphere are treated with more than the ordinary fullness. The earlier sections of this part are made short, each treating of a single subject, and the propositions are made to stand out prominently. [illegible] the close of each section are *Exercises* designed to illustrate and apply the principles contained in the section, rather than to extend the pupil's knowledge of geometrical facts. These features, together with the synopses at the close of the sections, practical teachers cannot fail to appreciate.

PART III., which is contained only in the *University Edition*, [illegible] been written with special reference to the needs of students in the *University of Michigan.* Our admirable system of public High-Schools, of which schools there is now one in almost every considerable village, promises ere long to become to us something near what the German Gymnasia are to their Universities. In order to promote the legitimate development of these schools, it is necessary that the University resign to them the work of instruction in the elements of the various branches, as fast and as far as they are prepared in sufficient numbers to undertake it. It is thought that these schools should now give the instruction in Elementary Geometry, which has hitherto been given in our ordinary college course. The first two parts of this volume furnish this amount of instruction, and students are expected to pass examination upon it on their entrance into the University. This amount of preparation enables students to extend their knowledge of Geometry, during the Freshman year in the University, considerably beyond what has hitherto been practicable. As a text-book for such students, *Part* III. has been written. At this stage of his progress, the student is prepared to learn to investigate for himself. Hence he is here furnished with a large collection of well classified theorems and problems, which afford a review of all that has gone before, extend his knowledge of geometrical truth, and give him the needed discipline in original demonstration. To develop the power of independent thought, is the most difficult, while it is the most important part of the teacher's work. Great pains have therefore been taken, in this part of the work, to render such aid, and *only such*, as a student ought to require in advancing from the stage in which he has been following the processes of others, to that of independent reasoning. In the second place, this part contains what is usually styled *Applications of Algebra to Geometry*, with an extended and carefully selected range of examples in this important subject. A third purpose has been to present in this part an introduction to what is often spoken of as the *Modern Geometry*, by which is meant the results of modern thought in developing geometrical truth upon the direct method. While, as a system of geometrical reasoning, this Geometry is not philosophically different from that with which the student of Euclid is familiar, and which is properly distinguished as the *special* or *direct* method) the character of the facts developed is quite novel. So much so, indeed, that the student who has no knowledge of Geometry but that which our common text-books furnish, knows absolutely [illegible] of the domain into which most of the brilliant advances of the present century have been made. He knows not even the terms in which the ideas of such writers as PONCELET, CHASLES, and SALMON, are expressed, and he is quite as much a stranger to the thought. In this part are presented the fundamental ideas concerning *Loci*, *Symmetry*, *Maxima* and *Minima*, *Isoperimetry*, the theory of *Transversals*, *Anharmonic Ratio*, *Polars*, *Radical Axes*, and other modern views concerning the circle.

PART IV. is *Plane* and *Spherical Trigonometry*, with the requisite Tables. While this Part, as a whole, is much more complete than the treatises in common use in our schools, it is so arranged that a shorter course can be taken by such as desire it. Thus, for a shorter course in Plane Trigonometry, see NOTE on page 55. In Spherical Trigonometry, the first three sections, either with or without the *Introduction* on Projection, will afford a very satisfactory elementary course.

A few words as to the manner in which this plan has been executed, may be important. In general, the *Definitions* are those usually given, with such slight alterations as have been suggested by reflection and experience. There are, however, a few exceptions. Among these is the definition of an *Angle*. I can but regard the attempt to define an angle as *The difference in direction between two lines*, or *The amount of divergence*, as needlessly vague, abstract, and perplexing to a student, as well as questionable on philosophical grounds. The definition given in the text will be seen to be, at bottom, the old one, the conception being slightly altered to bring it into more close connection with common thought, and also with the idea of an angle as generated by the revolution of a line. As to *Parallels*, and the definition of *similarity*, my experience as a teacher is decidedly in favor of retaining the old notions. So also in adopting a definition of a *Trigonometrical Function*, I am compelled to adhere to the geometrical conception. A ratio is a complex concept, and consequently not so easy of application as a simple one. For this reason, among others, I prefer the *differential* to the *differential coefficient*, in the calculus, and a *line* to a *ratio*, in Trigonometry. Moreover, I have found that students invariably rely upon the geometrical conception, even when first taught the other; hence I am not surprised that all our writers who define a trigonometrical function as a ratio, hasten to tell the pupil what it means, by giving him the geometrical illustrations. Nor are the superior facility which the geometrical conception affords for a full elucidation of the doctrine of the signs of the functions, and its admirable adaptation to fix these laws in the mind, considerations to be lost sight of in selecting the definition.

Surely no apology is needed, at the present day, for introducing the idea of *motion* into Elementary Geometry, notwithstanding the rigorous and disdainful manner with which its entrance was long resisted by the old Geometers. And, having admitted this idea, the conception of loci as generated by motion would seem to follow as a logical necessity. In like manner, I take it, the *Infinitesimal method* must come in. Its directness, simplicity, and necessity in applied mathematics, demand its recognition in the elements. In two or three instances, I have presented the *reductio ad absurdum*, where the methods are equivalents, and have always in presenting the infinitesimal method woven in the idea of *limits*, which I conceive to be fundamentally the same as the infinitesimal. Thus we bring the lower and higher mathematics into closer connection.

The *order of arrangement* in *Plane Geometry* (Chap. I.), is thought to be simple, philosophical, and practical. A glance at the table of contents will show what it is. This arrangement secures the very important result, that each section presents some particular *method of proof*, and holds the student to it, until it is familiar. True, it requires that a larger number of propositions be demonstrated from fundamental truths; but who will consider this an objection?

To such as consider it the sole province of geometrical demonstration, to convince the mind of the truth of a proposition, not a few theorems in these and ordinary pages must seem quite superfluous. To them, *Prop.* I., page 121, may afford some merriment. But those who, with myself, consider Geometry as a branch of practical logic, the aim of which is to detect and state the steps which actually lie between premise and conclusion, will see the propriety of such demonstrations; and for each individual of the other class, a separate treatise will be needed, since no two minds will intuitively grant exactly the same propositions.

To Ex-President Hill, of Harvard, I am indebted for the confirmation of an opinion which had been previously forming in my mind, that the study of Geometry as a branch of logic, should be preceded by a presentation of its leading facts. The works of Compagnon, Tappan, and our lamented countryman, Chauvenet, have been within reach during the entire work of preparation, and this volume would have been different, in some respects, if any one of these able treatises had not appeared before it.

In the preparation of Part III. the works of Rouché et Comberousse and Mulcahy have been freely used. For the very concise and elegant form in which the principle of Delambre, for the precise calculations of Trigonometrical Functions near their limits, is embodied in TABLE III., I am indebted to the recent work of President ELI T. TAPPAN, of Kenyon College, Ohio.

My long and intimate intercourse with Professor G. B. MERRIMAN, now of the department of Physics in the University, has been a source of great profit to me in the preparation of the entire work. His sound, practical judgment as a teacher of Geometry, and cultivated taste and skill as a Mathematician, have been ever at my service, and have done more than I can tell, in giving form to the work, both as respects its matter and its spirit.

EDWARD OLNEY.

UNIVERSITY OF MICHIGAN,
ANN ARBOR, *January*, 1872.

CONTENTS.

INTRODUCTION.

PAGE.

SECTION I.

LOGICO-MATHEMATICAL TERMS Defined and Illustrated............... 1-8

SECTION II.

THE GEOMETRICAL CONCEPTS Defined and Illustrated............... 8-11

PART I.

A FEW OF THE MORE IMPORTANT FACTS OF THE SCIENCE.

SECTION I.

ABOUT STRAIGHT LINES... 11-19

SECTION II.

ABOUT CIRCLES.. 19-25

SECTION III.

ABOUT ANGLES AND PARALLELS...................................... 25-30

SECTION IV.

ABOUT TRIANGLES...........

SECTION V.

ABOUT EQUAL FIGURES.. 35-39

SECTION VI.

ABOUT SIMILAR FIGURES, especially Triangles..................... 39-44

SECTION VII.

ABOUT AREAS.. 44-55

SECTION VIII.

ABOUT

PART II.

THE FUNDAMENTAL PROPOSITIONS OF ELEMENTARY GEOMETRY, DEMONSTRATED, ILLUSTRATED, AND APPLIED.

CHAPTER I.

PLANE GEOMETRY.

SECTION I.

PAGE.
PERPENDICULAR STRAIGHT LINES.......... 60–65

SECTION II.

OBLIQUE STRAIGHT LINES.......... 65–70

SECTION III.

PARALLELS.......... 70–77

SECTION IV.

RELATIVE POSITIONS OF STRAIGHT LINES AND CIRCUMFERENCES..... 78–85

SECTION V.

RELATIVE POSITIONS OF CIRCUMFERENCES.......... 86–93

SECTION VI.

MEASUREMENT OF ANGLES.......... 94–103

SECTION VII.

ANGLES OF POLYGONS AND THE RELATION BETWEEN THE ANGLES AND SIDES.

Of Triangles.......... 104–106
Of Quadrilaterals.......... 107–111
Of Polygons of more than Four Sides.......... 112–118
Of Regular Polygons.......... 118–120

SECTION VIII.

OF EQUALITY.

Of Lines and Circles.......... 121–122
Of Angles.......... 122–123
Of Triangles.......... 124–129
Of Quadrilaterals.......... 129–130
Of Polygons of more than Four Sides.......... 130–137

SECTION IX.

OF EQUIVALENCY AND AREAS.

Equivalency.......... 138–140
Area.......... 140–144

SECTION X.

PAGE.
OF SIMILARITY 144–153

SECTION XI.

APPLICATIONS OF THE DOCTRINE OF SIMILARITY TO THE DEVELOPMENT OF GEOMETRICAL PROPERTIES OF FIGURES.

Of the Relations of the Segments of two Lines intersecting each other and intersected by a circumference 153–154
Of the Bisector of an Angle of a Triangle 154–156
Areas of Similar Figures 156–158
Perimeters and the Rectification of the Circumference 158–160
Area of the Circle 160–163

CHAPTER II.

SOLID GEOMETRY.

SECTION I.

OF STRAIGHT LINES AND PLANES.

PAGE.
Perpendicular and Oblique Lines to a Plane 164–169
Parallel Lines and Planes 169–174

SECTION II.

OF SOLID ANGLES.

Of Diedrals 176–178
Of Triedrals 178–184
Of Polyedrals 185–186

SECTION III.

OF PRISMS AND CYLINDERS 187–199

SECTION IV.

OF PYRAMIDS AND CONES 199–209

SECTION V.

OF THE SPHERE.

Circles of the Sphere 210–211
Distances on the Surface of a Sphere 211–215
Spherical Angles 215–218
Tangent Planes 218–219
Spherical Triangles 219–226
Polar or Supplemental Triangles 226–229
Quadrature of the Surface of the Sphere 229–231
Lunes 231–235
Volume of Sphere 235–239

PART III.

AN ADVANCED COURSE IN GEOMETRY.

CHAPTER I.

EXERCISES IN GEOMETRICAL INVENTION.

SECTION I.

PAGE.
THEOREMS IN SPECIAL OR ELEMENTARY GEOMETRY................ 243–267

SECTION II.

PROBLEMS IN SPECIAL OR ELEMENTARY GEOMETRY................ 267–276

SECTION III.

APPLICATIONS OF ALGEBRA TO GEOMETRY........................ 276–288

CHAPTER II.

INTRODUCTION TO MODERN GEOMETRY.

SECTION I.

OF LOCI.. 288–296

SECTION II.

OF SYMMETRY.. 296–301

SECTION III.

OF MAXIMA AND MINIMA, AND ISOPERIMETRY..................... 302–306

SECTION IV.

OF TRANSVERSALS.. 306–310

SECTION V.

HARMONIC PROPORTION, AND HARMONIC PENCILS.................. 310–314

SECTION VI.

ANHARMONIC RATIO... 314–318

SECTION VII.

POLE AND POLAR IN RESPECT TO A CIRCLE...................... 319–323

SECTION VIII.

RADICAL AXES AND CENTRES OF SIMILITUDE OF CIRCLES.......... [illegible]

SPECIAL OR ELEMENTARY GEOMETRY.

INTRODUCTION.

SECTION I.

LOGICO-MATHEMATICAL TERMS.[*]

1. A Proposition is a statement of something to be considered or done.

Ill.—Thus, the common statement, "Life is short," is a proposition; so, also, we make, or state a proposition, when we say, "Let us seek earnestly after truth."—"The product of the divisor and quotient, plus the remainder, equals the dividend," and the requirement, "To reduce a fraction to its lowest terms," are examples of Arithmetical propositions.

2. Propositions are distinguished as *Axioms, Theorems, Lemmas, Corollaries, Postulates,* and *Problems.*

3. An Axiom is a proposition which states a principle that is so simple, elementary, and evident as to require no proof.

Ill.—Thus, "A part of a thing is less than the whole of it," "Equimultiples of equals are equal," are examples of axioms. If any one does not admit the truth of axioms, when he understands the terms used, we say that his mind is not sound, and that we cannot reason with him.

4. A Theorem is a proposition which states a real or supposed fact, whose truth or falsity we are to determine by reasoning.

Ill.—"If the same quantity be added to both numerator and denominator of a proper fraction, the value of the fraction will be increased," is a theorem. It is a statement the truth or falsity of which we are to determine by a course of reasoning.

---

[*] That is, terms used in the science in consequence of its logical character. The science of the Pure Mathematics may be considered as a department of practical logic.

5. *A Demonstration* is the course of reasoning by means of which the truth or falsity of a theorem is made to appear. The term is also applied to a logical statement of the reasons for the processes of a rule. A solution tells *how* a thing is done: a demonstration tells *why* it is so done. A demonstration is often called *proof*.

6. *A Lemma* is a theorem demonstrated for the purpose of using it in the demonstration of another theorem.

ILL.—Thus, in order to demonstrate the rule for finding the greatest common divisor of two or more numbers, it may be best first to prove that "A divisor of two numbers is a divisor of their sum, and also of their difference." This theorem, when proved for such a purpose, is called a *Lemma*.

The term *Lemma* is not much used, and is not very important, since most theorems, once proved, become in turn auxiliary to the proof of others, and hence might be called lemmas.

7. *A Corollary* is a subordinate theorem which is suggested, or the truth of which is made evident, in the course of the demonstration of a more general theorem, or which is a direct inference from a proposition, or a definition.

ILL.—Thus, by the discussion of the ordinary process of performing subtraction in Arithmetic, the following *Corollary* might be suggested: "Subtraction may also be performed by addition, as we can readily observe what number must be added to the subtrahend to produce the minuend."

8. *A Postulate* is a proposition which states that something can be done, and which is so evidently true as to require no process of reasoning to show that it is possible to be done. We may or may not know how to perform the operation.

11. A Solution is the process of performing a problem or an example. It should usually be accompanied by a demonstration of the process.

12. A Scholium is a remark made at the close of a discussion, and designed to call attention to some particular feature or features of it.

ILL.—Thus, after having discussed the subject of multiplication and division in Arithmetic, the remark that "Division is the converse of multiplication," is a scholium.

SYNOPSIS.

Subject of the section.
Proposition. *Ill.*
Varieties of propositions.
Axiom. *Ill.*
One who will not admit the truth of axioms.
Theorem. *Ill.*
Demonstration. Difference between a solution and a demonstration.
Lemma. *Ill.* Why the term is unimportant.
Corollary. *Ill.*
Postulate. *Ill.*
Problem. How stated. *Ill.*
Rule.
Solution.
Scholium. *Ill.*

◆

SECTION II.

THE GEOMETRICAL CONCEPTS.*

POINTS.

13. A Point is a place without size. Points are designated by letters.

ILL.—If we wish to designate any particular point (place) on the paper, we put a letter by it, and sometimes a dot on it. Thus, in *Fig.* 3, the ends of the line, which are points, are designated as "point A," "point D;" or, simply, as A and D. The points marked on the line are designated as "point B," "point C," or as B and C. F and E are two points above the line.

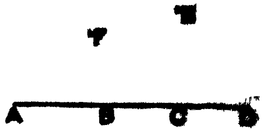


FIG. 3.

LINES.

14. *A Line* is the path of a point in motion. Lines are represented upon paper by marks made with a pen or pencil, the point of the pen or pencil representing the moving point. A line is designated by naming the letters written at its extremities, or somewhere upon it.

ILL.—In each case in *Fig.* 4, conceive a point to start from A and move along the path indicated by the mark to B. The path thus traced is a line. *Since a true point has no size, a line has no breadth*, though the marks by which we represent lines have some breadth. The first and third lines in the figure are each designated as "the line AB." The second line is considered as traced by a point starting from A and coming around to A again, so that B and A coincide. This line may be designated as the line A*mn*A, or A*mn*B. In the fourth case, there are three lines represented, which are designated, respectively, as A*m*B, A*n*B, and A*c*B; or, the last, as AB.

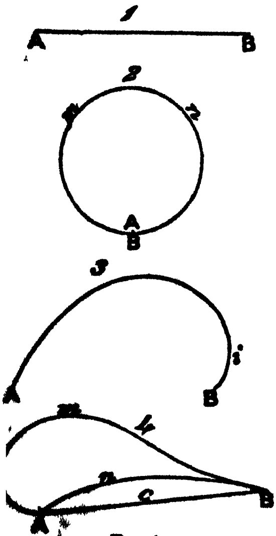


FIG. 4.

15. Lines are of ***Two Kinds***, *Straight* and *Curved*. A straight line is also called a *Right Line*. A curved line is often called simply a *Curve*.

16. *A Straight Line** is a line traced by a point which moves constantly in the same direction.

17. *A Curved Line* is a line traced by a point which constantly changes its direction of motion.

would say, that, starting from A, the point begins to move nearly north, but keeps changing its direction more and more toward the east, until at 8 it moves directly east; and from 8 it continues to change its course and moves more and more toward the south, till at *i* it is moving directly south. The same general truth is illustrated in 2 and 4, *Fig.* 4. The path of a ball thrown into the air, in any direction except directly up, represents a curved line. Most of the lines seen in nature are curved, as the edges of leaves, the shore of a river or lake, etc. Sometimes a path like that represented in *Fig.* 5 is called, though improperly, a *Broken Line*. It is not *a* line at all; that is, not *one* line: it is a series of straight lines.

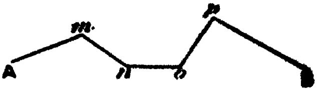


FIG. 5.

SURFACES.

18. ***A Surface*** is the path of a line in motion.*

19. Surfaces are of ***Two Kinds***, *Plane* and *Curved.*

20. ***A Plane Surface***, or simply a *Plane*, is a surface with which a straight line may be made to coincide in any direction. Such a surface may always be conceived as the path of a straight line in motion.

21. ***A Curved Surface*** is a surface in which, if lines are conceived to be drawn in all directions, some or all of them will be curved lines.

ILL'S.—Let AB, *Fig.* 6, be supposed to move to the right, so that its extremities A and B move at the same rate and in the same direction, A tracing the line AD, and B, the line BC. The path of the line, the figure ABCD, is a surface. This page is a surface, and may be conceived as the path of a line sliding like a ruler from top to bottom of it, or from one side to the other. Such a path will have length and breadth, being in the latter respect unlike a line, which has only length.

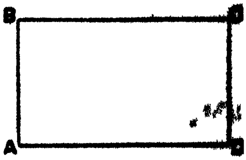


FIG. 6.

As a second illustration, suppose a fine wire bent into the form of the curve AmB, *Fig.* 7, and its ends A and B stuck into a rod, XY. Now, taking the rod XY in the fingers and rolling it, it is evident that the path of the line represented by the wire AmB, will be the surface of a ball (sphere).

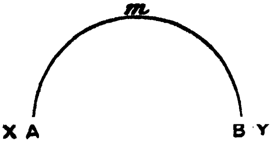


Fig. 7.

Again, suppose the rod XY be placed on the surface of this paper so that the wire A*m*B shall stand straight up from the paper, just as it would be if we could take hold of the curve at *m* and raise it right up, letting XY lie as it does in the figure. Now slide the rod straight up or down the page, making both ends move at the same rate. The path of A*m*B will be like the surface of a half-round rod (a semi-cylinder). Thus we see how surfaces plane and curved may be conceived as the paths of lines in motion.

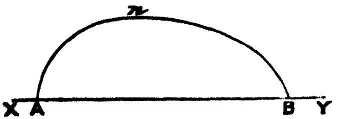


Fig. 8.

Ex. 1. If the curve A*n*B, *Fig.* 8, be conceived as revolved about the line XY, the surface of what object will its path be like?

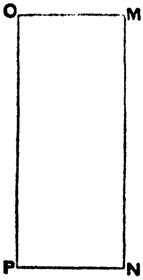


Fig. 9.

Ex. 2. If the figure OMNP, *Fig.* 9, be conceived as revolved about OP, what kind of a path will MN trace? What kind of paths will PN and OM trace?

Ans. One path will be like the surface of a joint of stove-pipe, *i. e.*, a cylindrical surface; and one will be like a flat wheel, *i. e.*, a circle.

Ex. 3. If you fasten one end of a cord at a point in the ceiling and hang a ball on the other end, and then make the ball swing around in a circle, what kind of a surface will the string describe?

[Note.—The student is not necessarily expected to give the geometrical name of the surface, but rather to tell in his own way what it is like, so as to make it clear that he conceives the thing itself.]

Ex. 4. If you were to draw lines in all directions on the surface of the stove-pipe, might any of them be straight? Could *all* of them be straight? What kind of a surface is this, therefore?

Ex. 5. Can you draw a straight line on the surface of a ball? On the surface of an egg? What kind of surfaces are these?

Ex. 6. When the carpenter wishes to make the surface of a board perfectly flat, he takes a ruler whose edge is a straight line, and lays this straight edge on the surface in all directions, watching closely to see if it always touches. Which of our definitions is he illustrating by his practice?

Ex. 7. When the miller wishes to make flat the surface of one of the large stones with which wheat is ground into flour, he sometimes takes a ruler with a straight edge, and smearing the edge with paint, applies it in all directions to the surface, and then chips off the stone where the paint is left on it. What principles is he illustrating?

Ex. 8. How can you conceive a straight line to move so that it shall not generate a surface?

ANGLES.

22. *A Plane Angle*, or simply an *Angle*, is the opening between two lines which meet each other. The point in which the lines meet is called the ***vertex*,** and the lines are called the ***sides*.** An angle is designated by placing a letter at its vertex, and one at each of its sides. In reading, we name the letter at the vertex when there is but one vertex at the point, and the three letters when there are two or more vertices at the same point. In the latter case, the letter at the vertex is put between the other two.

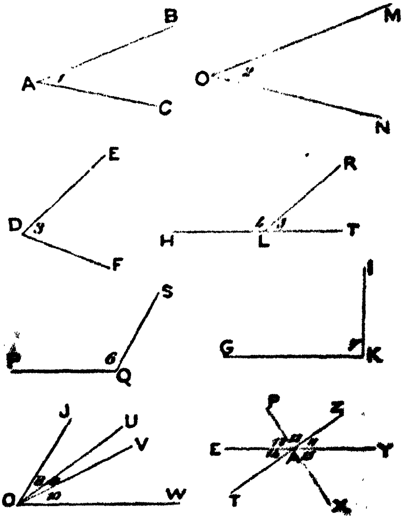


Fig. 10.

Ill.—In common language an angle is called a *corner*. The opening between the two lines AB and AC, in which the figure 1 stands, is called the angle A; or, if we choose, we may call it the angle BAC. At L there are two vertices, so that were we to say the angle L, one would not know whether we meant the angle (corner) in which 4 stands, or that in which 5 stands. To avoid this ambiguity, we say the angle HLR for the former, and RLT for the latter. The angle ZAY is the corner in which 11 stands; that is, the opening between the two lines AY and AZ. In designating an angle by three letters, it is immaterial which letter stands first so that the one at the vertex is put between the other two. Thus, PQS and SQP are both designations of the angle in which 6 stands. An angle is also frequently designated by putting a letter or figure in it and near the vertex.

23. ***The Size of an Angle*** **depends upon the rapidity with which its sides separate, and not upon their length.**

Ill.—The angles BAC and MON, *Fig.* 10, are equal, since the sides separate at the same rate, although the sides of the latter are more prolonged than those of the former. The sides DF and DE separate faster than AB and AC, hence the angle EDF is greater than the angle BAC.

24. ***Adjacent Angles*** are angles so situated as to have a common vertex and one common side lying between them.

Ill.—In *Fig.* 10, angles 4 and 5 are adjacent, since they have the common vertex L, and the common side LR. Angles 9 and 10 are also adjacent, as are also 8 and 9.

25. Angles are distinguished as *Right Angles* and *Oblique Angles*. Oblique angles are either *Acute* or *Obtuse*.

26. ***A Right Angle*** is an angle included between two straight lines which meet each other in such a manner as to make the adjacent angles equal. ***An Acute Angle*** is an angle which is less than a right angle, *i. e.*, one whose sides separate less rapidly. ***An Obtuse Angle*** is an angle which is greater than a right angle, *i. e.*, one whose sides separate more rapidly.

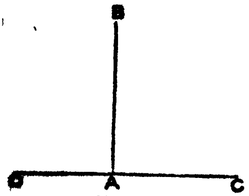


Fig. 11.

Ill.—As in common language an angle is called a *corner*, so a right angle is called a *square corner*; an acute, a *sharp corner*; and an obtuse angle might be called a blunt corner. In *Fig.* 11, BAC and DAB are right angles. In *Fig.* 10, 1, 2, 3, 5, 8, 9, and 10 are acute angles, 4 and 6 are obtuse, and 7 is a right angle.

A SOLID.

27. ***A Solid*** **is a limited portion of space. It may also be conceived as the path of a surface in motion.**

Ill.—Suppose you have a block of wood like that represented in *Fig.* 12, with all its corners (angles) square corners (right angles). Hold it still in your

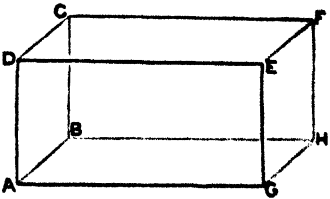


Fig. 12.

fingers a moment, and fix your mind upon it. Now take the block away and think of the space (place) where it was. This space will be of just the same form as the block of wood, and by a little effort you can think of it just as well as of the wood. This *space* is an example of what we call a *Solid* in Geometry. In fact, the solids of Geometry are not solids at all in the common sense of solids; they are only just *places of certain shapes.*

Again, hold your ball still a moment in your fingers and then let it drop, and think of the place it filled when you had it in your fingers. It is this *place*, shaped just like your ball, that we think about, and talk about as a *solid*, in Geometry.

In order to see how a solid may be conceived as the path of a surface, suppose you cut out a piece of paper of just the same size as the end of the block represented in Fig. 12. Let ABCD represent this piece of paper. Now, holding the paper in a perpendicular position, as ABCD is represented in the figure, move it along to the right, so that its angles shall trace the lines AC, BH, DE, and CF. When the paper has moved to the position CHFE, its path will be just the same space as the block of wood occupied. This path, or the space through which the surface represented by the piece of paper moved, is the solid.

Ex. 1. If a semicircle is conceived as revolved around its diameter, what is the path through which it moves? See *Fig.* 7.

Ex. 2. If the surface OMNP, *Fig.* 9, is conceived as revolved around OP, what is the path through which it moves?

CAUTION.—The student needs to be careful and distinguish between the *surface* traced by the *line* MN, and the *solid* traced by the *surface* OMNP.

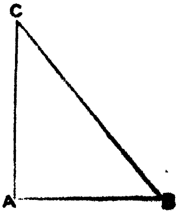


Fig. 13.

Ex. 3. If the surface represented by ABC be conceived as revolved about its side CA, what kind of a solid is its path?

[NOTE.—As has been said before, the student is not necessarily expected to *name* these solids, but rather to show, in his own language, that he has the conception.]

Ex. 4. As you fill a vessel with water, what is the solid traced by the surface of the water?

Ans. The same as the space within the vessel.

Ex. 5. If a circle is conceived as lying horizontally, and then moved directly up, what will be the solid described, *i. e.*, its path? Do not confound the surface described with the solid. What describes the surface? What the solid?

EXTENSION AND FORM.

28. Extension means a stretching or reaching out. Hence, a *Point* has no extension. It has only position (place). A *Line* stretches or reaches out, but only in length, as it has no width. Hence, a line is said to have *One Dimension,* viz., length. A *Surface* extends not only in length, but also in breadth; and hence has *Two Dimensions,* viz., length and breadth. A *Solid* has *Three Dimensions,* viz., length, breadth, and thickness.

ILL.—Suppose we think of a point as capable of stretching out (extending) in one direction. It would become a line. Now suppose the line to stretch out (extend) in another direction—to widen. It would become a surface. Finally, suppose the surface capable of thickening, that is, extending in another direction. It would become a solid.

29. The *Limits* (extremities) of a line are points.
The *Limits* (boundaries) of a surface are lines.
The *Limits* (boundaries) of a solid are surfaces.

30. Magnitude (size) is the result of extension. Lines, surfaces, and solids are the geometrical magnitudes. A point is not a magnitude, since it has no size. The magnitude of a line is its length; of a surface, its area; of a solid, its volume.

31. Figure or Form (shape) is the result of position of points. The form of a line (as straight or curved) depends upon the relative position of the points in the line. The form of a surface (as plane or curved) depends upon the relative position of the points in it. The form of a solid depends upon the relative position of the points in its surface. Lines, surfaces, and solids are the geometrical figures.*

ILL.—In *Fig.* 14, it is easy to conceive the form of the lines by knowing the position of points in the lines. By taking a quantity of common pins of different lengths, sticking them upright in a board, and conceiving the heads to represent points in a surface, we can readily see how the position of the points in a surface determine its form.

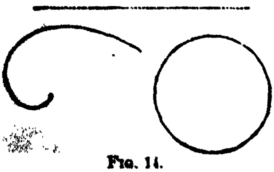


FIG. 14.

Ex. 1. Suppose a line to begin to contract in length, and continue the operation till it can contract no longer, what does it become? That is, what is the minor limit of a line?

---

* Lines, surfaces, and solids are called magnitudes when reference is had to their *extent*, and figures when reference is had to their *form*.

Ex. 2. If a surface contracts in one dimension, as width, till it reaches its limit, what does it become? If it contracts to its limit in both dimensions, what does it become?

Ex. 3. If a solid contracts to its limit in one dimension, what does it pass into? If in two dimensions? If in three dimensions?

Ex. 4. What kind of a surface is that, every point in which is equally distant from a given point?

32. Geometry treats of *magnitude* and *form* as the result of extension and position.

The *Geometrical Concepts* are points, lines, surfaces (including plane and spherical angles), and solids (including solid angles).

The *Object* of the science is the measurement and comparison of these concepts.

Plane Geometry treats of figures all of whose parts are confined to one plane. *Solid Geometry*, called also *Geometry of Space*, and *Geometry of Three Dimensions*, treats of figures whose parts lie in different planes. The division of Part II. into two chapters is founded upon this distinction. In the *Higher* or *General* Geometry these divisions are marked by the terms "*Of Loci in a Plane*," and "*Of Loci in Space*."

SYNOPSIS.

GEOMETRICAL CONCEPTS.

- POINT....
    - What.—How designated.—*Ill.*
    - Dimensions of.
    - Limit of Line.—Surface.—Solid.
- LINE.....
    - What.
    - How designated.
    - Dimensions of.
    - Limit of Surface.
    - Kinds
        - Straight.—What.—*Ill.*
        - Curved.—What.—*Ill.*
        - Broken (?)
- SURFACE..
    - What.
    - Dimensions of.
    - Limit of Solid.
    - Kinds
        - Plane.—What.—*Ill.*
        - Curved.—What.—*Ill.*
    - Angle
        - What.—Size depends on what.—Adjacent.
        - Kinds
            - Right.—What.—*Ill.*
            - Oblique
                - Acute.—What.—*Ill.*
                - Obtuse.—What.—*Ill.*
- SOLID.....What.—*Ill.*—Examples.

GEOMETRY..

- Treats of
    - Magnitude.—What.—Result of what.
    - Figure or form.—What.—Result of what.
- Concepts.—What.
- Object.—What.

PART I.

A FEW OF THE MORE IMPORTANT FACTS OF THE SCIENCE.

SECTION I.

ABOUT STRAIGHT LINES.

33. Prob.*—To measure a straight line with the dividers and scale.*

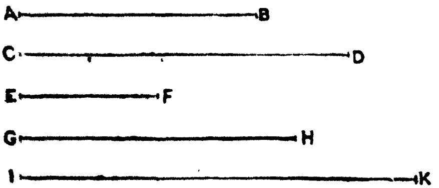


FIG. 15.

SOLUTION.—Let AB, *Fig.* 15, be the line to be measured. Take the dividers, *Fig.* 2 (frontispiece), and placing the sharp point A firmly upon the end A of the line AB, open the dividers till the other point B (the pencil point) just reaches the other end of the line B. Then letting the dividers remain open just this amount, place the point A on the lower end of the left hand scale, as at *o*, *Fig.* 1, and notice where the point B reaches. In this case it reaches 3 spaces beyond the figure 1. Now, as this scale is inches and *tenths* of inches,* the line AB is 1.3 inches long.

Ex. 1. What is the length of CD? *Ans.* .15 of a foot.

Ex. 2. What is the length of EF? *Ans.* .75 of an inch.

Ex. 3. What is the length of GH? *Ans.* 1¼ inches.

Ex. 4. What is the length of IK? *Ans.* .18 of a foot.

Ex. 5. Draw a line 3 inches long.

Ex. 6. Draw a line 2.15 inches long.

Ex. 7. Draw a line 1.25 inches long.

Ex. 8. Draw a line .85 of an inch long.

---

* The next scale to the right is divided into 10ths and 100ths of a foot. Thus from *p* to *K* [illegible] of a foot, and the smaller divisions [illegible]

[NOTE.—Suppose a fine elastic cord were attached by each of its ends to the points A and B of the dividers; when they were opened so as to reach from C to D, *Fig.* 15, the cord would represent the line CD. Now applying the dividers to the scale is the same as laying this cord on the scale. Without the cord, we can imagine the distance between the points of the dividers to be a line of the same length as CD.]

Ex. 9. Find in the same way as above the length and width of this page. Also the distance from one corner (angle) to the opposite one (the diagonal).

34. *Prob.—To find the sum of two lines.*

SOLUTION.—To find the sum of AB and CD, I * first draw the indefinite line E*x*. With the dividers I obtain the length of AB, by placing one point on A and extending the other to B. This length I now lay off on the indefinite line E*x*, by putting one point of the dividers at E and with the other marking the point F. EF is thus made equal to AB. In the same manner taking the length of CD with the dividers, I lay it off from F on the line F*x*. Thus I obtain EG=EF + FG=AB + CD. Hence, the sum of AB and CD is EG.

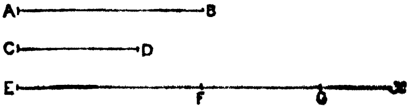


FIG. 16.

[NOTE.—The student may measure EG by (*33*) and find the sum of AB and CD in inches or feet; but it is most important that he be able to look upon EG as the sum itself.]

Ex. 1. Find the sum of AB and EF, *Fig.* 15.

Ex. 2. Find the sum of EF, CD, and GH, *Fig.* 15.

Ex. 3. Make a line twice as long as CD, *Fig.* 16. Three times as

35. *Prob.—To find the difference of two lines.*

SOLUTION.—To find the difference of AB and CD, I take the length of the less line AB with the dividers; and placing one point of the dividers at one extremity of CD, as C, make C*e* = AB. Then is *e*D the difference of AB and CD, since *e*D = CD − C*e* = CD − AB.

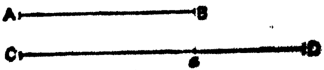


FIG. 17.

1. Find the difference of IK and EF, *Fig.* 15.

Ex. 2. Find the difference of GH and CD, *Fig.* 15.

Ex. 3. Find how much longer IK, *Fig.* 15, is than the sum of EF, *Fig.* 15, and CD, *Fig.* 16.

Ex. 4. Find the difference of the sum of AB and CH, and the sum of CD and EF, *Fig.* 15.

36. *Prob.*—*To compare the lengths of two lines; that is, to find their ratio (approximately*).*

SOLUTION.—To compare the lengths of AB and CD, I lay off AB, the shorter, upon CD, as C*a*. (If AB could be applied two or more times to CD, I should apply it as many times as CD would contain it.) Now I apply the remainder of CD, viz., *a*D, to AB, as many times as AB will contain it, which is once with the remainder *b*B. This remainder I now apply to *a*D, and find it contained once with a remainder *c*D. Again, I apply this last remainder to *b*B, and find it contained twice with a remainder *d*B. This last remainder I now apply to *c*D, and find it contained 3 times, without any remainder. This last measure, *d*B, is a common measure of the two lines. Calling *d*B 1, I now observe that

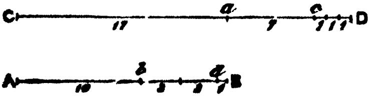


FIG. 18.

$$\begin{aligned}
cD &= 3dB = 3;\\
bd &= 2cD = 6;\\
ac &= bB = bd + dB = 7;\\
aD &= ac + cD = 10;\\
AB &= Ab + bB = aD + ac = 17;\\
CD &= Ca + aD = AB + aD = 27.
\end{aligned}$$

Hence the lines AB and CD are to each other as the numbers 17 and 27; AB is $\frac{17}{27}$ of CD; or, expressed in the form of a proportion, AB : CD : : 17 : 27.

[NOTE.—This process will be seen to be the same as that developed in Arithmetic and Algebra for finding the greatest or highest Common Measure of two numbers, and should be studied in connection with a review of those processes. See COMPLETE ARITHMETIC (*115*), and COMPLETE SCHOOL ALGEBRA (*137*).]

Ex. 1. Find, as above, the approximate ratio of AB to CD, *Fig.* 15.
Ratio, 13 : 18.

Ex. 2. Find, as above, the approximate ratio of CD and IK, *Fig.* 15.
Ratio, 5 : 6.

---

* This method does not get the exact ratio, because of the imperfection of measurement, and also because lines are sometimes incommensurable, as will appear hereafter.

Ex. 3. Find, as above, the approximate ratio of EF to GH, *Fig.* 15.
Ratio, 1 : 2.

Ex. 4. Find, as above, the approximate ratio of EF to CD, *Fig.* 15.
Ratio, 5 : 12.

---

37. To Intersect is to cross; and a crossing is called an *Intersection.*

38. To Bisect anything is to divide it into two *equal* parts.

39. Prob.—*To bisect a given line.*

SOLUTION.—To bisect the line AB, I take the dividers; and opening them so that the line between their points is more than half as long as AB, I place the sharp point A on the point A, and holding it firmly there, make a little mark with the pencil point B, as nearly as I can guess, opposite the middle of the line. Then, being careful to keep the dividers open just the same, I place the sharp point on B, and make a mark intersecting the first one, as at *m*. Now, doing just the same on the other side of the line, I make two marks intersecting each other, as at *n*. Finally, I draw a line from *m* to *n*, and where this line crosses AB is its middle point; that is, AO is equal to OB. [Why this is so we do not propose to tell now. The student needs only to learn how to do it. He should *measure* AO and OB, and thus test the accuracy of his work.]

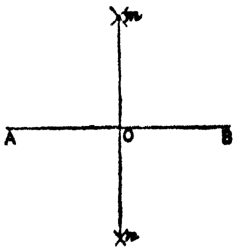


FIG. 19.

Ex. 1. Is it necessary that the dividers be opened just as wide when the marks are made through *n*, as when they are made through *m*? Try it.

Ex. 2. Suppose you make the marks through *m* as directed, but, in making those through *n*, you have the dividers *wider* open when you put the point on A than when you put it on B; will the line joining *m* and *n* then cross AB in the middle? If not, on which side of the middle will O be? Try it.

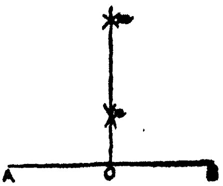


FIG. 20.

Ex. 3. Can you bisect a line by making the marks all on one side of it? If so, do it.

40. *Axiom.*—A straight line is the shortest path between two points.*

Ill.—If a cord is *stretched* across the table, it marks a straight line. In this way the carpenter marks a straight line. Having rubbed a cord, called a chalk-line, with chalk, he *stretches it tightly* from one point to another on the surfaces upon which he wishes to mark the line, and then raising the middle of the cord, lets it snap upon the surface. So the gardener makes the edges of his paths straight by *stretching* a cord along them. These operations depend upon the principle that when the line between the points is the shortest possible, it is straight.

41. *Axiom.—Two points in a straight line determine its position.*

Ill.—If the farmer wants a straight fence built, he sets two stakes to mark its ends. From these its entire course becomes known. This is the principle upon which aligning (or sighting) depends. Having given two points in the required line, by looking in the direction of one from the other, we look along a straight line, and are thus able to locate other points in the line. If the points A and B are marked, by putting the eye at A and looking steadily towards B, we can tell whether D and E are in the same straight line with A and B, or not. So we can observe that C′ and C″ are *not* in the line; but that C is. This process of discovering other points in a line with two given points is called aligning, or sighting. In this way a row of trees is made straight, or a line of stakes set. It is the principle upon which the surveyor runs his lines, and the hunter aims his gun. In the latter case, the two sights are the given points, and the mark, or game, is a third point, which the marksman wishes to have in the same straight line as the sights.

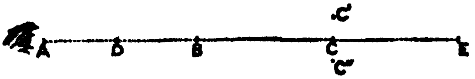


Fig. 21.

42. *Axiom.—Between the same two points there is one straight line, and only one.*

Ill.—Let any two letters on this page represent the situation of two points; we readily see that there is one, and only one, straight path between them. Again, let a corner of the desk represent one point and a corner of the ceiling of the room represent another point; we perceive at once that, if a point is conceived to pass in a straight line from one to the other, it will always trace the same path. In short, as soon as two points are mentioned, we think of the distance between them as a single straight line,—for example, the centre of the earth and the centre of the sun.

---

* An axiom may be *illustrated*, but it needs no *demonstration*. We may explain the terms used and elaborate the condensed statement; but if, when its meaning is clearly understood, any one does not grant the *truth* of its statement, he has not a sound mind, and we cannot reason with him.

Once more, conceive A and B, *Fig.* 21, to be two points in the path of a point moving from A in the direction of B. Now *all* the points in the same direction from A as B is, are in this path; and any point out of this line, as C′ or C″, is in a different direction from A.

In this manner we draw a straight line on paper by laying the straight edge of a ruler on two points through which we wish the line to pass, and passing a pen or pencil along this edge.

COR.—*Two straight lines can intersect in but one point; for, if they had two points common, they would coincide and not intersect.*

Ex. 1. A railroad is to be run from the town A to town B. If it is made straight, through what points will it pass? Can it pass through any points not in the same direction from A as B is?

Ex. 2. If I live on the south side of a straight railroad, and my friend on the north side, but five miles farther east, and two miles farther north, and the road from my house to his is straight, how many times does it cross the railroad?

Ex. 3. Can you always draw a straight line which shall cut a curve (whatever curve it may be) in two points? Try it.

Ex. 4. Detroit is directly east of where I live. How could I drive my horse there and never turn his head to the east? Would he have to travel in straight lines or in a curve? If I drive him on a curve, how can I manage it so that his head will be east for but an instant? If his head is all the time east, what is the line in which I drive him?

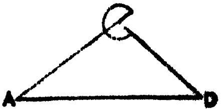


FIG. 22

SUG.—The figure will suggest how the first may be accomplished.

43. *A Perpendicular* to a given line is a line which makes a right angle (**26**) with the given line. The latter is also perpendicular to the former. ***Oblique Lines*** are such as are not perpendicular to each other, and which meet if sufficiently extended.

ILL.—In *Fig.* 11, BA is perpendicular to DC; so also AC is perpendicular to BA. In *Fig.* 10, KC and KI are perpendicular to each other. The other lines in *Fig.* 10 are oblique to each other.

44. Prob.—*To erect a perpendicular to a given line at a given point in the line.*

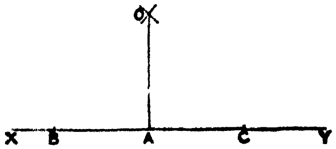


FIG. 23.

SOLUTION.—Suppose I want to erect a perpendicular to the line XY, at the point A. With the dividers I measure off a distance AB on one side of the point A, and *an equal distance* AC on the other side. Then opening the dividers a little wider, I put the sharp point on B and make a mark with the pencil point, as at O, about where I think the perpendicular will go. Then, keeping the dividers *open just the same*, I put the sharp point on C, and make a mark intersecting the former one at O. Now, drawing a line through O and A, it is the perpendicular sought.

Ex. 1. Suppose I make a mistake and close up the dividers a little after making the first mark through O, and then make the second mark; which way will the line lean? Will it be a perpendicular or an oblique line in this case? What kind of an angle would OAY be? What OAX? What kind of angles are these when OA is a perpendicular?

Ex. 2. Suppose I should mistake a point nearer to A than B was taken, and use it as I did C, having the dividers open just alike when I made the two marks through O; which way would the line lean (incline)? (Same questions as in the last.)

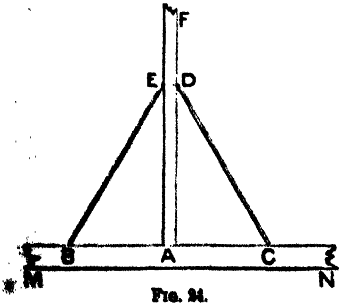


FIG. 24.

ILL.—A carpenter wishes to get the piece of timber AF at right angles to MN, into which it is mortised at A. So he measures off AB and AC, equal distances from A; and taking two poles of equal length (say 10 feet long), has the end of one held steadily at B and the end of the other at C, and moves (racks, as he calls it) the end F to the right or left until the ends E and D of the poles are exactly opposite, as in the figure. AF is then perpendicular to MN.

45. Prob.—*From a point without a given line, to draw a perpendicular to the line.*

SOLUTION.—I wish to draw a perpendicular from O to the line XY. I first open the dividers wide enough, so that when I place the sharp point on O the pencil will mark the line XY in two points, as B and C, when it swings around. Marking these two points, I put the sharp point first on B and afterward on C, *keeping them open just* alike in both cases, and make the two marks intersecting at D. Placing the straight edge of the ruler on the points O and D, I draw the line OA along its edge. OA is the perpendicular required.

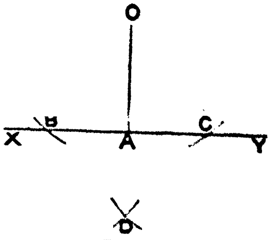


FIG. 25.

EX. 1. Let fall a perpendicular from a point, as O, upon a straight line, as XY, without making any marks on the opposite side of XY from O.

EX. 2. A mason wishes to build a wall from O, in the wall AB, "straight across" (perpendicular) to the wall CD, which is 8 feet from AB. He has only his 10-foot pole, which is subdivided into feet and inches, with which to find the point in the opposite wall at which the cross wall must join. How shall he find it?

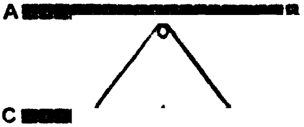


FIG. 26.

SECTION II.

ABOUT CIRCLES.

46. ***A Circle*** is a plane surface bounded by a curved line every point in which is equally distant from a point within.

47. ***The Circumference*** of a Circle is the curved line every point in which is equally distant from a point within.

48. ***The Centre*** of a Circle is the point within, which is equally distant from every point in the circumference.

49. ***An Arc*** is a part of a circumference.

50. ***A Radius*** is a line drawn from the centre to any point in the circumference of a Circle.

51. ***A Diameter*** of a Circle is a line passing through the centre and terminating in the circumference.

around? What will the string represent? What is the surface passed over by the string?

Ex. 3. If you take the dividers, *Fig.* 1, and open them (say 2 inches), and then place the sharp point, A, firmly on the paper while *you turn them around, making the pencil point, B, mark the paper as it goes, what kind of a line will be described? What is the line joining the points of the dividers?* What line describes the circle? If the dividers only turn a little way, what is the line* ibed?

4. If a boy skating on the ice makes a curve which bends where just alike, what kind of a path will he make? Does the boy describe a circle? How might you conceive the circle inclosed by his path, as described? Is a circle described by a point or by a line?

[NOTE.—The word "circle" is used in common language as equivalent to "circumference." It is also thus used in General Geometry. But, however the words may be used, the pupil should be taught to mark the distinction between the plane surface inclosed and the bounding line.]

Ex. 5. In how many points can a straight line intersect a circumference? In how many points can one circumference intersect another?

Ex. 6. There is a piece of ground in the form of a circle, the radius of which is 100 rods, by which run two roads; one road runs within 80 rods of the centre, and the other within 100 rods. How do the roads lie with reference to the ground?

Ex. 7. When you unwind a thread by drawing it off a spool in the ordinary way, what geometrical line does the unwound thread represent?

Ex. 8. In a circle whose diameter is 50 feet, there are drawn two chords, one is 20 feet long, and the other 30 feet. Which is nearer the centre?

Ex. 9. There are two circles whose radii are respectively 12 and 18 feet. The distance from the centre of one to the centre of the other is 25 feet. Do the circumferences intersect? Would they intersect if the centres were 3 feet apart? How would they lie in reference to each other in the latter case? How if their centres were 30 feet apart? How if they were 35 feet apart?

---

* The line [illegible]
points of the

Ex. 11. If you draw two chords in the same circle, one of which is twice as long as the other, will the arc cut off by the longer chord be twice as long as the arc cut off by the shorter? Will it be more than twice as long, or less?

55. Theorem.—*The chord of a sixth part of the circum, of a circle is just equal to the radius of the same circle.*

ILL.—If I draw a circle, and then, being careful not to open or close the dividers, place the sharp point on the circumference at some point, as A, and mark the circumference at another point, as B, with the pencil point, and then move the sharp point to B and mark again, as C, I find that when I have measured off six such chords, each equal to the radius, I return exactly to A, the point of starting.

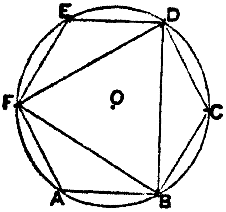


FIG. 81.

Moreover, if I draw the chords AB, BC, etc., I have a regular figure with six equal sides. A figure with six sides is called a hexagon. This hexagon is called *regular*, because its sides are equal each to each, and its angles are also mutually equal.

Again, if I unite the alternate angles of the regular hexagon, as FB, BD, and DF, I have a regular triangle, called an equilateral triangle.

56. Inscribed Figures are figures drawn *in* a circle, and having the vertices of all their angles in the circumference, as the hexagon and triangle in the last illustration. When the figure is without, and all its sides touch but do not cut the circumference, it is *circumscribed* about the circle.

Ex. 1. Draw a regular hexagon whose side is two inches.

Ex. 2. Inscribe an equilateral triangle in a circle whose radius is one inch.

57. Prob.—*To find the centre of a circle when the circumference is drawn* (or, as we usually say, *known*).

SOLUTION.—The circumference of my circle is drawn, but the centre is not

58. Prob.—*To pass a circumference through three given points.*

SOLUTION.—I wish to pass a circumference through the three given points A, B, and C. [The pupil should first designate three points by dots on his paper, slate, or board, and then proceed according to the solution.] In order to do this, I join A and B with a line, and also B and C. I now bisect these lines with the perpendiculars MN and RS, as in the last problem. The intersection of these perpendiculars, O, is the centre of the required circle. Now setting the sharp point of the dividers upon O and opening them till the pencil point just reaches A (B or C will answer as well), I draw the circumference with O as its centre and the radius OA, and find that it passes through the three given points A, B, and C.

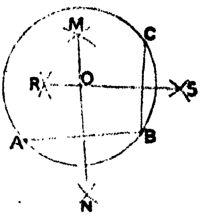


FIG. 33.

Ex. 1. To pass a circumference through the three vertices of a *triangle, i. e., to circumscribe a circumference about a triangle, as this operation is technically called.*

SUG.—This is just like the last, A, B, and C being the vertices of the triangle. The four figures in the margin represent the successive steps in the solution. First draw the given triangle. Then take the first step in the solution, then the second, etc.

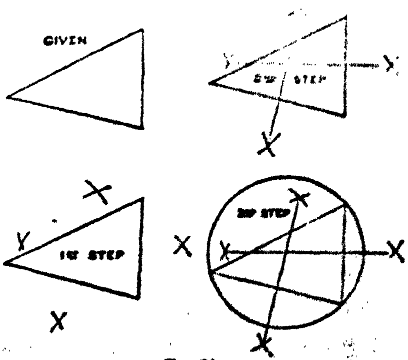


FIG. 34.

Ex. 2. Given the centre of a circle and a point in the circumference, to draw the circle.

SUG.—Make a dot on the board to indicate the centre, and another dot to indicate the point in the circumference to be found. This is what is given. You are

Ex. 1. Draw an acute angle and also an obtuse angle, and then compare them as above.

Ex. 2. Draw a small acute angle and a large acute one, and then compare them as above.

Ex. 3. Draw a small acute angle, and then draw another angle 3 times as large.

Ex. 4. Draw an acute angle, and also a right angle, and compare them as above.

SUG.—Article (*39*) shows how to draw a right angle.

Ex. 5. Draw any angle, and then draw another equal to it.

Ex. 6. *Show that the angles a, b, and c are respectively* $\frac{1}{3}$, $\frac{2}{3}$, *and .6 of a right angle.**

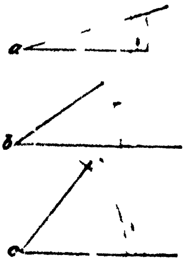


FIG 38

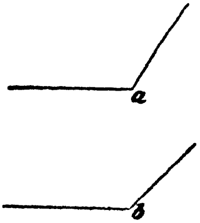


FIG 39

Ex. 7. Show that angles *a* and *b*, *Fig.* 39, are respectively $1\frac{1}{3}$ and $1\frac{1}{2}$ times a right angle.

Ex. 8. Draw a regular inscribed hexagon, as in *Fig.* 31, and then comparing any one of its angles with a right angle, find that it is $1\frac{1}{3}$ times a right angle.

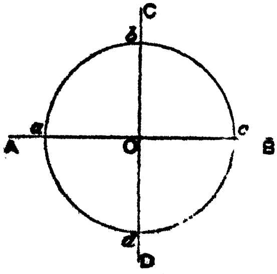


FIG. 40.

Ex. 9. Draw an equilateral triangle, as in *Fig.* 31, and find that any angle of it is $\frac{2}{3}$ of a right angle.

Ex. 10. Show that a right angle is measured by $\frac{1}{4}$ of a circumference.

SOLUTION.—If CD is perpendicular to AB, the four angles formed are equal, and each is a right angle. But, as all of them taken together are measured by the whole circumference, one of them is measured by $\frac{1}{4}$ of the circumference.

---

* Of course, absolute accuracy is not to be expected in such solutions.

62. *An Inscribed Angle* is an angle whose vertex is in the circumference of a circle, and whose sides are chords, as A, *Fig.* 41.

63. *Theorem.*—*An inscribed angle is measured by one-half the arc included between its sides.*

ILL.—The meaning of this is that an inscribed angle like A, which includes any particular arc, as *cd*, is only half as large as an angle would be at the centre, as *c*O*d*, whose sides included the same arc, *cd*, or an equal arc. Thus, in this case, drawing the arc *ab* from A as a centre, with the same radius, O*d*, as *cd* is drawn with, I find that *ab* which measures A is ½ of *cd* which measures *c*O*d*.

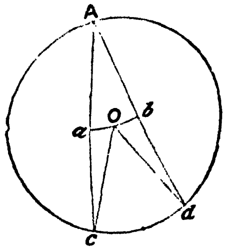


FIG. 41.

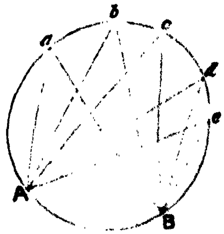


FIG. 42.

Ex. 1. Which of the angles *a*, *b*, *c*, *d*, *e* is the largest? What is *a* measured by? What *b*? What *c*? What *d*? What *e*? *Fig.* 42.

Ex. 2. Which is the greatest angle, *a*, *b*, or *c*, *Fig.* 43? By what is *a* measured? By what *b*? By what *c*? What is the measure of a right angle? [See Example 10 in the preceding set.]

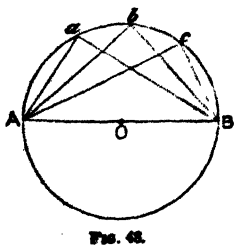


FIG. 43.

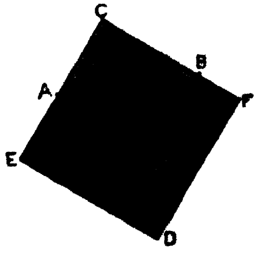


FIG. 44.

Ex. 3. Suppose I take a square card like CEDF, with a hole in one corner as at C, and sticking two pins firmly in my paper, as at A and B, place the corner of the card between them, as in *Fig.* 44, and keeping the sides of the card snug against the pins, put a pencil through the hole C and move it around to A and then back to B; what kind of a line will the pencil trace? Will it make any difference whether C is a right angle or not? If any difference, what?

Ex. 4. By what part of a circumference is an angle of a regular inscribed hexagon measured? See (*55*), and *Fig.* 31. How many right angles is the angle of the hexagon equal to? What is the sum of the six angles equal to? *Ans. to last,* 8 right angles.

Ex. 5. Show, from the way in which an equilateral triangle is constructed in *Fig.* 31, that one of its angles is measured by ⅙ of a circumference, and hence is ⅔ of a right angle.

---

64. Theorem.—*When two lines intersect, they form either four right angles, or two equal acute and two equal obtuse angles.*

ILL.—[The pupil can illustrate this for himself by drawing lines and noticing what angles are equal.]

Ex. 1. Having a carpenter's square, an instrument represented by MON, I wish to test the angle O and ascertain whether it is, as it should be, a right angle. I draw an indefinite right line AB, and placing the angle O at some point C on this line with ON extending to the right on CB, I draw a line along OM. Turning the square over so that ON shall lie on CA, I draw another line along OM. Three cases may occur.—1st. Suppose the first line drawn along OM is CF, and the second CE; what kind of an angle is O? 2d. Suppose the first line drawn is CE and the second CF; what kind of an angle is O? 3d. Suppose the first and second lines drawn along OM coincide and are CD; what kind of an angle is O?

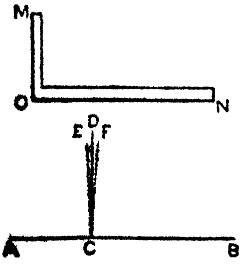


FIG. 45.

Ex. 2. Show that the sum of all the angles formed by drawing lines on one side of a given line, and to the same point in the line, is two right angles.

---

65. Prob.—*To bisect a given angle.*

SOLUTION.—I wish to divide the angle AOB into two equal parts, *i. e.,* to bisect it. With O, the vertex, as a centre, and any convenient radius, as O*a*, I strike an arc, as *ba*, cutting the sides of the angle. Then from *a* and *b* as centres, with the same radius in each case, I strike two arcs intersecting as at P. Drawing a line through P and O, it bisects the angle; *i. e.*, the angle POA = angle BOP. [Let the pupil try this by cutting out the angle AOB, and then folding the paper along the line P, or cutting it through in the line OP, and then putting one angle on the other, and thus see if they do not fit.]

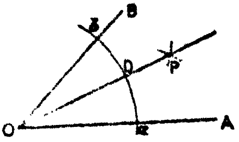


FIG. 46.

Ex. 1. Draw an angle equal to ½ of a right angle.

SUG.—First draw a right angle and then bisect it.

Ex. 2. Draw an angle equal to ⅓ of a right angle.

SUG.—Draw a circle. Inscribe an equilateral triangle. [Do it neatly, by rule, as in (*55*).] Then bisect any angle of this triangle. This will be ⅓ of a right angle, since the whole angle is ⅔. See Ex. 9 (*61*).

Ex. 3. How does it appear that the angle EDF, *Fig.* 31, is ½ of a right angle?

66. Parallel Straight Lines are such as, lying in the same plane, will not meet how far soever they are produced either way.

ILL.—The sides of this page are parallel lines, as are also the top and bottom. The lines in *Fig.* 47 are parallel.

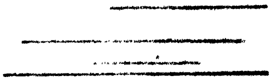


FIG. 47.

67. Prob.*—To draw a line through a given point and parallel to a given line.*

SOLUTION.—I wish to draw a line through the point O and parallel to the line AB. [The pupil should first draw some line, as AB, and mark some point, as O.] I take O as a centre, and with a radius* greater than the shortest distance to AB, as O*a*, draw an indefinite arc *a*P. Then with *a* as a centre, and the same radius, I draw an arc from O to the line AB at *b*. Taking the distance O*b* (the chord) in the dividers, I put the sharp point on *a* and strike a small arc intersecting this indefinite arc, as at P. Finally, drawing a line through O and P, it is the parallel sought.

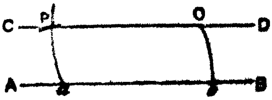


FIG. 48.

---

* This means "put the sharp point of the dividers on O and open them till the distance between the points (the radius) is more than the distance from O to AB."

68. Theorem.—*Two parallel lines are everywhere the same distance apart.*

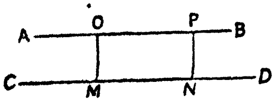


Fig. 49.

Ill.—Let AB and CD be two parallel lines. I will examine them at the two points O and P. To find how far apart the lines are at these points I draw the perpendiculars OM and PN. [The pupil should not *guess at these, but actually draw them as instructed in (44).] Measuring these, I find them equal.*

We can understand that this proposition must be true, since the lines could not approach each other for awhile and then separate more and more without being crooked; or, if they kept on approaching each other, they would meet after awhile, and so not be parallel.

---

69. Theorem.—*Parallel lines make no angle with each other.*

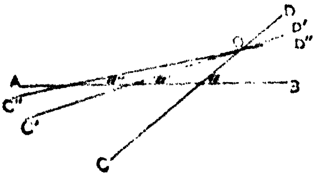


Fig. 50.

Ill.—Let AB be a straight line, and suppose CD another straight line passing through the point O. Now let CD turn around, first into the position D'C', then into D''C'', etc., all the time passing through O. It is evident that the angle which this line makes with the line AB is all the time growing less, *i. e.*, $a' < a$, and $a'' < a'$. It is also evident that this angle will become 0 when the lines become parallel; for it becomes less and less all the time, but is always something so long as the lines are not parallel.

---

70. Theorem.—*Parallel lines have the same direction with each other.*

Ill.—Thus, in *Fig.* 47, the parallel lines all extend to the right and left, *i. e.*, in the same direction.

Ex. 1. How shall the farmer tell whether the opposite sides of his farm are parallel?

Ex. 2. If we wish to cross over from one parallel road to another, is it of any use to travel farther in the hope that the distance across will be less?

Ex. 3. If a straight line intersects two parallel lines, how many angles are formed? How many angles of the same size? May they *all* be of the same size? When? When will they not be all of the same size?

SECTION IV.

ABOUT TRIANGLES.

71. *A Plane Triangle,* or simply *A Triangle, is* a plane *figure bounded by three straight lines.*

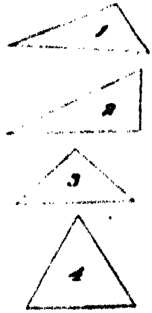


FIG. 51.

72. With respect to their sides, triangles are distinguished as *Scalene, Isosceles,* and *Equilateral.* A scalene triangle has no two sides equal. An isosceles triangle has two sides equal. An equilateral triangle has all its sides equal.

73. With respect to their angles, triangles are distinguished as *acute* angled, *right* angled, and *obtuse* angled. An acute angled triangle has three acute angles. A right angled triangle has *one* right angle, and the side opposite the right angle is called the *hypotenuse.* An obtuse angled triangle has *one* obtuse angle.

Ex. *Fig.* 51 affords illustrations of all the different kinds of triangles. Let the pupil point them out until he is perfectly familiar with the terms. He should also practise drawing the different kinds of triangles, for the purpose of familiarizing the names applied to the different kinds.

---

74. Theorem.—*The sum of the angles of a triangle is two right angles.*

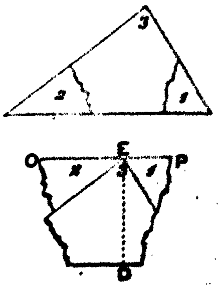


FIG. 52.

ILL.—Cut out any triangle from a piece of paper. Then cut off two of the angles, as 1 and 2, and turn them about and place them by the side of the other angle, as in the lower figure. You will then see that *the line* OP *is straight,* and that the three angles of the triangle just make up the two right angles OED and PED.

Ex. 1. If one angle of a triangle is a right angle, what is the sum of the other two?

Ex. 2. Can a triangle have more than *one* right angle? If two of its angles were right angles, what would the third angle be?

Ex. 3. Can a triangle have more than one obtuse angle?

Sug.—Try and see if you can draw a triangle with two right angles, or two obtuse angles.

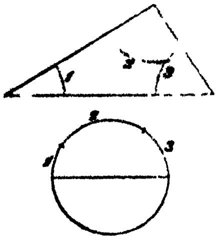


FIG. 53.

Ex. 4. Construct any triangle, and draw arcs measuring its angles. Then draw a circle *with the same radius* as the one used to measure the angles, and lay off upon the circumference the arcs measuring the angles. The sum of these arcs will always make up just a semi-circumference. What does this show?

Ex. 5. If two angles of one triangle are equal to two angles of another, can the third angles be unequal? Why?

---

75. Prob.—To make two triangles just alike.

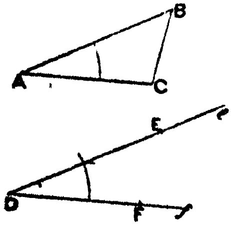


FIG. 54.

Solution.—There are three ways of doing this:

1st Way.—Suppose I have any triangle, as ABC, and want to make another just like it. I first draw an arc measuring any one of the angles, as A, of the given triangle. Then I make an angle D equal to the angle A, and draw the sides D*e* and D*f*. Now I measure DE = AB, and DF = AC. If I now draw EF, the triangle DEF will be just like ABC, so that, were I to cut them out, I could apply one like a pattern to the other, and it would just fit.

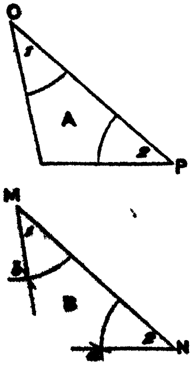


FIG. 55.

2d Way.—I have a triangle A, and wish to make another just like it. I draw arcs measuring any two of its angles, as O and P. Then, making a line MN equal to OP, I make an angle at M equal to O, and one at N, on the same side of MN, equal to P. Now making these two sides M*b* and N*a* long enough to meet (or, as we say, "producing them till they meet"), I have a second triangle, B, just like the first triangle, A. Were I to cut out the first triangle, it would fit on the second just like a pattern.

3d Way.—I have a triangle ACB, and want to make another just like it. I make a line DE equal to some side of the given triangle, as AB. Then taking AC as radius, I describe an arc from D as a center, and in like manner, with BC as radius and E as a center, describe another arc. Through the intersections of these arcs, as F, I draw DF and EF. The triangle DEF is just like ABC. [Try it by drawing as described, and then cutting out one triangle, and seeing if you cannot fit it as a pattern on the other.]

Ex. 1. In any triangle, which side is opposite the greatest angle? Which opposite the least angle?

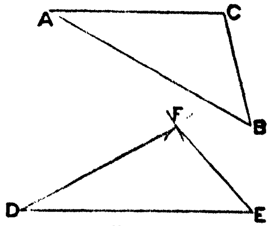


Fig. 56.

Ex. 2. If you have two triangles with an angle in each equal, but the sides about this angle longer in one triangle than in the other, can you make one fit on the other as a pattern? Cut out two such triangles and try it.

Ex. 3. Can you make a triangle so that one of its sides shall be as long as both the others, or longer than both?

Ex. 4. Can you make a triangle so that one of its sides shall be less than the difference between the other two, or equal to the difference?

Ex. 5. If you have two triangles with *only* one side and one angle in the one equal to one side and one angle in the other, can you apply one as a pattern and make it fit on the other? Cut out two such triangles and try it.

Ex. 6. If you have two triangles with *only* two sides of one respectively equal to two sides of the other, can you make one fit as a pattern on the other? Try it.

Ex. 7. If you have two triangles with two sides in one equal respectively to two sides in the other, and the included angle in one greater than in the other, how is it with the third sides of the triangles?

76. Theorem.—The lines which bisect the angles of a triangle meet within the triangle at a common point.

Ill.—Try it, by drawing a triangle, and then bisecting its angles, as taught in (65). You will need to do it very neatly, or the lines will not meet. It is a delicate operation. Try it in various forms of triangles, as equilateral, right angled, scalene, obtuse angled, etc.

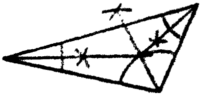


Fig. 57.

***77. Theorem.**—The lines drawn from the vertices of a triangle to the middle of the opposite sides meet in a common point within the triangle.*

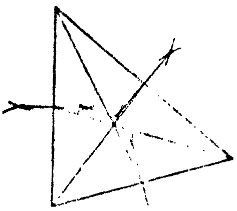


Fig. 58.

ILL.—Draw a triangle. Bisect each of the sides as taught in (*39*). Then join each angle and the middle of its opposite side with a straight line. If you do the work well, the three lines will cross each other at a common point within the triangle.

***78. Theorem.**—The perpendiculars which bisect the sides of a triangle meet at a common point, which may be within or without the triangle, or in one of its sides, according to the form of the triangle.*

ILL.—Draw *an acute angled triangle*, and bisect its sides by perpendiculars. If you do it with accuracy, they will meet at a common point *within* the triangle.

Draw an *obtuse angled triangle*, bisect its sides with perpendiculars, and they will meet at a common point *without* the triangle.

Draw a *right angled triangle*, and the perpendiculars will meet in the side opposite the right angle (the hypotenuse).

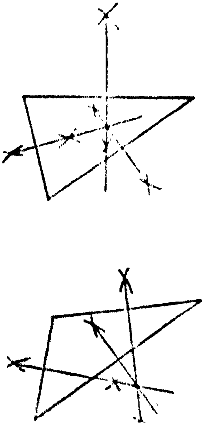


Fig. 59.

Ex. 1. Draw an *equilateral* triangle, and find the three points characterized in the last three articles. Are they all in one place, or are they in different places?

Ex. 2. Draw a scalene triangle, and find the three points as above. Are they all in the same place, or are they in different places?

***79. Prob.**—To inscribe a circle in a given triangle.*

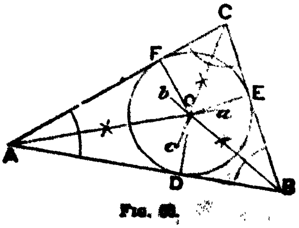


Fig. 60.

SOLUTION.—I wish to inscribe a circle in the triangle ABC; that is, a circle to which the sides of the triangle shall be tangents. [First draw the triangle.] I bisect the angles as taught in (*65*); and then from the point O, where these intersect, I let fall perpendiculars upon the sides, as taught in (*45*). Then from O as to one of these perpendiculars (they are all equal), I draw a circle, and it is the circle required.

80. ***Prob.***—*To circumscribe a circle about a given triangle.*

Solution.—I wish to circumscribe a circle about the triangle ABC which I have drawn. To do this, I *bisect the sides* with perpendiculars, and find their common intersection O, as taught in (*78*). With O as a centre and a radius equal to OB, the distance from O to the vertex of any one of the angles, as these distances are all equal, I draw a circle. This is the circumscribed circle, that is, the circle in whose circumference the vertices of the triangle lie. [This is really the same as Prob. (*58*).]

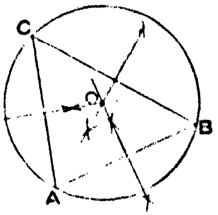


Fig. 61.

SECTION V.

ABOUT EQUAL FIGURES.

81. ***Equal,*** in geometry, signifies alike in all respects, *i. e.*, of the same shape and the same size.

82. ***Equivalent*** figures are such as have the same area, *i. e.*, are of the same size, irrespective of their form.

Ex. 1. Can a triangle be *equal* to a circle? Can it be *equivalent?* Can a circle be equivalent to a square? Can it be equal to a square?

Ex. 2. Can a right angled triangle be equal to an equilateral triangle? Can a right angled triangle be equal to an isosceles triangle? If either is possible, construct figures illustrating it.

83. ***Prob.***—*To apply one straight line to another.*

Solution.—[*Applying* figures to each other is a very important thing in geometry, and may seem a little curious at first; but it is, in reality, very simple. The pupil must become perfectly familiar with it.] We will first apply the line AB to the equal line CD. Take the line AB,* and placing the end A upon the end C of the line CD, make the line AB take the same direction as CD, and put the former upon the latter. Now, since the lines are equal, the

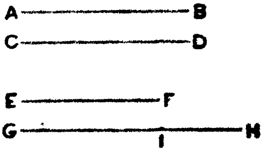


Fig. 62.

---

* That is, *think* about it just as if it were a little rod which you could pick up and handle.

extremity (or the point) B will fall upon D, and the two lines will coincide throughout their whole extent.

Again, we will apply the line EF to the line GH. Taking the line EF (*think* of it as a little rod which you can pick up and handle), put the point E upon G, and making the line EF take the same direction as GH, put the former upon the latter. Now, since EF is shorter than GH, the point (extremity) F will fall somewhere on the line GH, as at I. Therefore the lines do not coincide throughout their whole extent, and are not equal.

84. *Prob.—To apply one plane angle to another.*

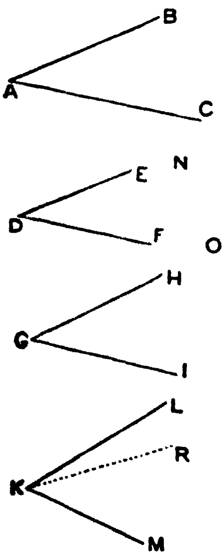


Fig. 63.

Solution.—First we will apply one angle to another equal angle. Thus, to apply BAC to the equal angle EDF. Take the angle BAC (*think* of it as if it were two little rods put firmly together at this angle, and so that you could pick them up and handle them), and placing the vertex (point) A upon the vertex (point) D, make the side AC take the *direction* DF. As AC happens to be longer than DF, the extremity C will fall beyond F, at some point, as O. But we do not care for this, as the size of an angle does not depend upon the length of the sides. Now, while A lies on D, and the line AC on DF, let the line AB be conceived as lying in the plane of the paper also (*i. e.*, on it). Since the angle BAC is equal to EDF, the line AB will take the direction DE, and will fall on it, though the point B will fall somewhere beyond E, as at N, as AB chances to be longer than DE. The two angles therefore coincide, and are equal. [Notice carefully just what is meant by saying that the angles are equal. We do not mean that the sides are of the same length, but that the *opening* between them is the same, *i. e.*, that one is just as sharp a corner as the other.]

Queries.—If BAC were greater than EDF, and we should begin by putting A upon D, and make AC fall upon DF, where would AB fall, without the angle EDF or within it? If BAC were less than EDF, and we proceed as before, placing the vertex A on D, and AC on DF, would AB fall without EDF or within it?

Again, let us attempt to apply the angle HGI to LKM. Placing the vertex G on the vertex K, making the side GI take the direction KM, and then bringing GH into the plane of the paper, the side GH will fall within the angle LKM (as in the line KR), since the angle HGI is less than LKM. The angles, therefore, do not coincide.

85. Prob.—*When two triangles have two sides and the included angle of one equal to two sides and the included angle of the other, to apply one triangle to the other.*

SOLUTION.—In the two triangles ABC and DEF, let the angle A be equal to the angle D, the side AB = the side DE, and AC = DF. We will apply the triangle ABC* to DEF. Take the triangle ABC and place the vertex A upon the vertex D, making the side AC take the direction DF. Since AC = DF, the extremity C will then fall on F.† Now bring the triangle ABC into the plane of DEF, keeping AC in DF, and the line AB will take the direction DE, since the angle A = the angle D. Again, as AB = DE, the extremity B will fall upon E. Thus we have placed ABC upon DEF, so that A falls upon D, C upon F, and B upon E, and find that they exactly coincide.

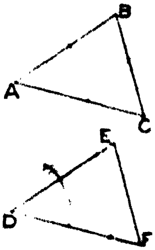


FIG. 64.

Ex. 1. Suppose you attempt to apply ABC in the last figure to DEF by placing B on D, and letting BC fall upon DF. Where will C fall? Measure it and find out. Which side will then fall nearly or quite on DE? Will it fall exactly on it? On which side will it fall? Can you make the triangles coincide (fit) in this way?

Ex. 2. Can you make the triangles in the last figure coincide by placing C upon D, and letting CA fall upon DF? Where will A fall? What line will fall on or near DE? Will it fall without DE, or within?

Ex. 3. Construct two isosceles triangles,‡ as ACB and DEF, in which AC = CB = DE = EF. Can you apply DEF to ABC by putting D upon A? Describe the process. Can you put D upon A and DE upon AB, and make the triangles coincide? Can you make the triangles coincide by putting F upon A?

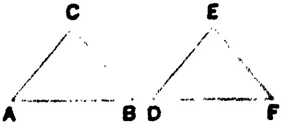


FIG. 65.

If so, describe the process. Can you make them coincide by putting E upon A? If not, point out the difficulties.

---

* Think of ABC as

in the manner described.

† It will make it clearer if the pupil thinks of ABC, at this stage of the

AC on DF, but the angle B

d *afterward* bring the triangle ABC

AC on the line DF.

ng drawn, and that accurately, according to

rule.

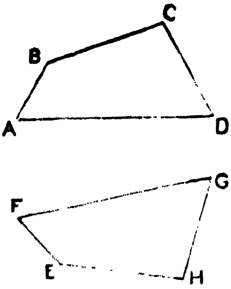


FIG. 66.

Ex. 4. Construct two equal trapeziums,* as ABCD and EFCH, and describe the process of applying one to the other.

SOLUTION.—I will apply EFCH to ABCD. As the angle E is equal to the angle B, I will begin by putting the vertex E on B, and making EH fall upon BC. Since $EH = BC$, H will fall on C. Now, as angle $H =$ angle C, HC will take the direction (fall on) CD; and since $HC = CD$, C will fall on D. Again, as $C = D$, CF will take the direction DA; and since $CF = DA$, F will fall on A. Finally, as $F = A$, FE will take the direction AB; and since $FE = AB$, E will fall on B, as it ought, since I started by conceiving E as placed on B.

Ex. 5. Describe the application of ABCD in the last figure to EFCH, by beginning with C upon H.

Ex. 6. Having two equal equilateral triangles, can you apply one to the other by beginning indifferently with any one angle of one upon any one angle of the other? Draw two such triangles, and go through with the details of the application.

---

86. Prob.—*Given two triangles with two angles and the included side of the one respectively equal to two angles and the included side of the other, to apply one triangle to the other.*

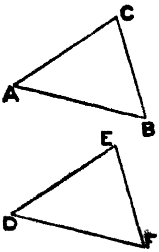


FIG. 67.

SOLUTION.—[The pupil should first draw any triangle, as ABC. Then make a line DF equal to AB, and at the extremities D and F make angles, as D and F, respectively equal to A and B. This is preliminary.] Having the two triangles ABC and DEF, in which $A = D$, $B = F$, and $AB = DF$, I propose to apply one to the other. I will apply ABC to DEF. Taking ABC, I place A upon D, and make AB take the direction and fall upon DF. Since $AB = DF$, B will fall upon F. Now keeping the line AB in DF, I conceive the triangle ABC to come into the plane of DEF. Since $A = D$, the side AC will take the direction DE, and the extremity C of AC will fall somewhere in the line DE, or in DE produced. Also, since $B = F$, the line BC will take the direction FE, and the extremity C of BC will fall somewhere in FE or FE produced. Finally, as C falls in DE and

---

* The teacher must insist upon the figures being drawn, and that accurately, according to rule.

FE both, it must be at **E**, their intersection. Thus I find that the triangle **ABC**, when applied to **DEF**, coincides with it throughout.

Ex. 1. Given the two triangles **DEF** and **ABC**, in which **DE** = **AB**, **D** = **A**, but **E** > **B**; show how an attempt to apply one to the other fails.

Solution.—Since angle **D** = angle **A**,* I apply the vertex **D** to the vertex **A**, and make **DE** take the direction **AB**. As **DE** = **AB**, **E** will fall on **B**, and the sides **DE** and **AB** will coincide. Again, since **D** = **A**, the side **DF** will take the direction **AC** when the planes of the triangles coincide, and the extremity **F** will fall in **AC**, or in **AC** produced (really in **AC** produced, in this case). Finally, since **E** > **B**, **EF** will fall to the right of **BC**, and the application fails.

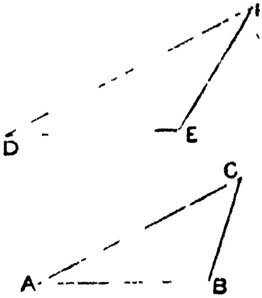


Fig. 68.

Ex. 2. Construct two trapeziums with their respective sides equal, as **AC** = **HE**, **AB** = **HG**, **BD** = **GF**, and **CD** = **EF**, but with their angles unequal; and show how an attempt to apply one to the other fails.

Ex. 3. If the sides of two trapeziums, as in the last figure, are equal, and two of the angles including a side in one are respectively equal to the corresponding angles in the other, as **A** = **H**, and **B** = **G**, can one be applied to the other? If so, give the details of the process.

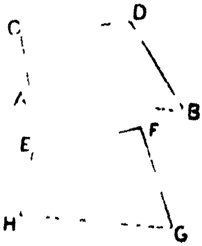


Fig. 69.

SECTION VI.

ABOUT SIMILAR FIGURES, ESPECIALLY TRIANGLES.

87. ***Similar Figures*** are such as are shaped alike—*i. e.*, have the same form.

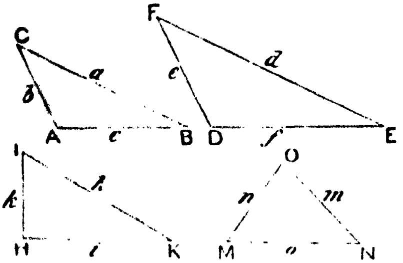
A more scientific definition is, *Similar Figures* are such as have their angles respectively equal, and their homologous (corresponding) sides proportional.

---

* Be careful to distinguish between the vertex, which is a point, and the angle, which is the *opening* between the lines.

88. ***Homologous,*** *or* ***Corresponding Sides*** of similar figures *are those which are included between equal angles in the respective figures.*

IN SIMILAR TRIANGLES, THE HOMOLOGOUS SIDES ARE THOSE OPPOSITE THE EQUAL ANGLES.



ILL.—The triangles **ABC** and **DEF** are similar, for they are of the same shape. But it is easy to see that **ABC** is not similar to **IHK** or **MON**. The pupil should notice that **A** = **D**, **C** = **F**, and **B** = **E**. Also, side e is 1⅓ times b, side f is 1⅓ times c, and side d is 1⅓ times a; so that $f : c :: e : b$, and $f : c :: d : a$, and $d : a :: e : b$. Now there are no such relations existing between the parts of **ABC** and **IHK**. The angles **B** and **K** are nearly equal, but **A** is much larger than **H**, and **C** is smaller than **I**. So these triangles are *not* mutually equiangular, *i. e.*, each angle in one has not an equal angle in the other. Again, as to their sides, **IH** is a little less than **AC**, but **HK** is greater than **AB**. These two triangles are, therefore, not similar.

In the similar triangles **ABC** and **DEF**, b is homologous with e, since they are opposite the equal angles **B** and **E**. For a like reason a is homologous with d, and c with f. It may also be observed, that the shortest sides in two similar triangles are homologous with each other; the longest sides are also homologous with each other, and the sides intermediate in length are homologous with each other.

Ex. 1. Can a scalene triangle be similar to an isosceles triangle? Can an obtuse angled triangle be similar to a right angled triangle?

Ex. 2. Are all squares similar figures?

SUG.—First, are the angles equal? Second, is any one side of one square to some side of another square as a second side of the first is to a second side of the second, etc.?

Ex. 3. A farmer has two fields, each of which has 4 sides and 4 right angles. The first field is 20 rods by 50, and the other 40 by 80. Are they similar?

SUG.—Are they mutually equiangular? Then are the lengths in the same ratio as the widths? If they are not similar, how long would the second have to be in order to make them similar? Draw two such figures, and see if they look alike in shape.

89. ***Prob.****—To find a fourth proportional to three given lines.*

SOLUTION.—I have the three given lines A, B, and C, and wish to find a fourth line such that

A shall be to B as C is to the *fourth line*, *i. e.*,

A : B : : C : *fourth line*.

To do this, I draw two indefinite lines OX and OY, from a common point O. On one of these, as OX, I lay off Oa = A, and Oc = B. Then on the other I make Ob = C, and draw ab. Finally, drawing a parallel to ab through the point c (**67**), I have Od as the line sought. Thus, calling Od, D, the proportion is

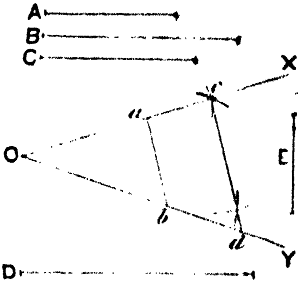


FIG. 71.

Oa : Oc : : Ob : Od, or

A : B : : C : D.

N.B.—*The order in which the lines are taken, and the way of drawing the lines ab and cd, are essential.* The following directions will insure correctness: *Lay off the* FIRST *and* SECOND *on the* SAME LINE, as on OX; *and the* THIRD *on the* OTHER LINE, as on OY. *Then join the extremities of the* FIRST *and* THIRD, *and draw the parallel through the extremity of the* SECOND.

Ex. 1. Show that if the order of the proportionals in *Fig.* 71 is B : A : : C : *fourth line*, the fourth proportional is E, *Fig.* 71.

Ex. 2. Show that a fourth proportional to A, B, and C is D. Also, that a fourth proportional to C, A, and B is E. Show that, if the order be A : C : : B : *fourth line*, D is still the fourth proportional. Show that B : C : : A : 2C, nearly.

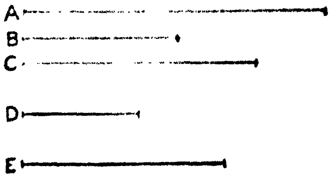


FIG. 72.

Ex. 3. Solve the proportion 3 : 8 : : 5 : x, and find x **geometrically.**

SUG.—Using the scale of 100ths of a foot, the figure is that in the margin. OD is the fourth proportional, or x = OD, which is found *by measurement* to be 13⅓, as it is by arithmetic.

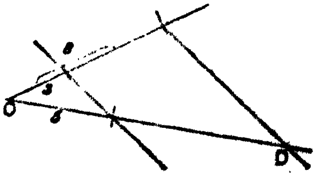


FIG. 73.

90. Prob.—*To draw a triangle similar to a given triangle, and having a given side.*

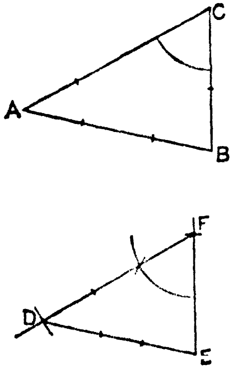


Fig. 74.

Solution.—*1st Method.*—I have a triangle ACB, and want to make another similar to it, but having the side homologous to BC equal to *a*. I draw an indefinite line, and on it take EF, equal to *a*. Then at F I make an angle equal to C, and make the side indefinite. Now I find a fourth proportional to BC, EF, and AC. Having found this, as in the last article, I lay it off from F, as FD. Drawing DE, I have DEF, the triangle required.

I can readily satisfy myself that DEF is similar to ABC, for besides the fact that it looks as if it were of the same shape, by measuring the other two angles, I find that E = B, and D = A. Moreover, I know that BC, EF, AC, and DF are proportional, because I made them so. And, by finding a fourth proportional to BC, EF, and AB, I find it exactly equal to DE. In like manner constructing a fourth proportional to AC, DF, and AB, I find it to be DE. So that the two triangles are mutually equiangular, and have their homologous sides (those opposite the equal angles) proportional. Hence, the triangles are similar.

2d Method.—But an easier way to construct DEF is to make the angle F = C as before, and then make E = B, and produce the sides till they meet in D. The triangles will then be similar, and the proportionality of the sides can be tested.

Ex. 1. Given a triangle whose sides are 7, 11, and 15, to construct a similar triangle having the side corresponding to the one which is 11 in the given triangle, 8.

Ex. 2. Construct two triangles with equal angles, and then compare the sides, and see whether you can make two triangles whose angles shall be respectively equal, and their sides not be proportional.

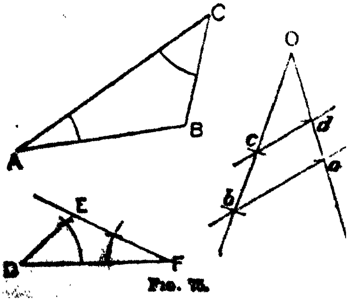


Fig. 75.

1.—Having the triangle ABC, make DEF equiangular with it, and then compare the homologous sides. In the figure D is made equal to C, and F to A; whence E = B. DE and BC are homologous sides, because opposite the F and A. DF gous with **AC**, because it is opposite angle **E**, which equals **B**. For a similar reason **EF** is homologous with **AB**. Now, taking two sides of **ABC**, as **BC** and **AB**, and a side of **DEF** homologous with one of them, as **DE**, and finding a fourth proportional **O*c***, it will be found exactly equal to **EF**; so that

BC : DE :: AB : EF (= O*c*).

Ex. 3. Make two triangles, two of whose angles shall be, one ¾ and the other ¼ of a right angle; but make the side included between these angles twice as great in the second triangle as in the first. What will be the ratio of the side opposite the angle ¾ in the first triangle to the homologous side in the second? What the relation of the sides opposite the angles ¼?

Ex. 4. If you make one triangle whose sides are 5, 8, and 3; and a second whose sides are 15, 24, and 9, will they be mutually equiangular? Which angles are the equal ones? Which are the homologous sides?

Ex. 5. There are three pairs of similar triangles in *Fig.* 76. Can you point them out? Also point out their homologous parts. Are all the triangles which you can make out from the figure similar to each other?

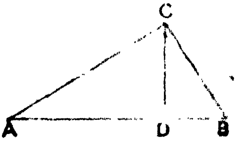


Fig. 76.

Ex. 6. Wishing to know the height **EC** of a house, I set up a stake **DB** 5 feet long; and putting my eye close to the ground, I moved back from the stake to **A**, so that the top of the stake and the top of the house were just in range (in a line). Then by measuring I found **AB** = 10 feet, and **AC** = 80 feet. What was the height of the house?

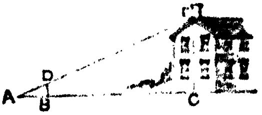


Fig. 77.

Ex. 7. If you take three sticks of different lengths and put them together by joining their ends two and two, so as to represent a triangle; can you, by putting together the same sticks in a different order, make a triangle of different form from the first? Will the angles opposite the same sticks always be the same?

Ex. 8. If you take more than three sticks (say 4), and make of them the boundary of a figure, by putting their ends together two and two, can you put them together so as to make another figure of different form? Can you make figures having different angles?

Ex. 9. If you take three sticks, **A** 3 inches long, **B** 5 inches, and C 6 inches; and also three other sticks, D 9 inches long, E 15 inches, and F 18 inches;* can you place them together so as to make dissimilar triangles? Will the corresponding angles of the two triangles be equal however you may arrange the sticks? If the sides of two triangles are proportional, will their angles be equal and the triangles similar?

Ex. 10. If you take four sticks, A 3 inches long, B 5 inches, C 6 inches, and K 4 inches; and also four other sticks, D 9 inches long, E 15 inches, F 18 inches, and L 12 inches;* can you place them together so as to make four-sided figures which shall be dissimilar (*i. e.*, not of the same shape)? Will the corresponding angles of the two figures be necessarily equal? If the sides of a four-sided figure are proportional, does it follow that the corresponding angles are equal, and the figures similar?

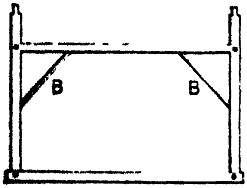


Fig 78

Ex. 11. Why do the braces in the frame of a building stiffen it? Is a four-sided figure stiff? *i. e.*, are its angles incapable of change while its sides remain of the same length? Can the angles of a triangle be changed while the sides remain unchanged?

◆

SECTION VII.

ABOUT AREAS.

91. ***A Quadrilateral*** is a plane surface inclosed by *four straight* lines.

92. There are three *Classes* of quadrilaterals, viz., ***Trapeziums***, ***Trapezoids***, and *Parallelograms*.

93. ***A Trapezium*** is a quadrilateral which has no two of its sides parallel to each other.

94. ***A Trapezoid*** is a quadrilateral which has but two of its sides parallel to each other.

---

* Notice that the sides are proportional, *i. e.*, in the same ratio taken two and two.

95. ***A Parallelogram*** is a quadrilateral which has its opposite sides parallel.

96. ***A Rectangle*** is a parallelogram whose angles are right angles.

97. ***A Square*** is an equilateral rectangle.*

98. ***A Rhombus*** is a parallelogram whose angles are not right angles, and all of whose sides are equal.

99. ***A Rhomboid*** is a parallelogram whose angles are not right-angles, and two of whose sides are greater than the other two.

Ill.—The figures in the margin are all quadrilaterals. A is a trapezium. (Why?) B is a trapezoid. (Why?) C, D, E, and F are parallelograms. (Why?) D and E are rectangles, although D is the form usually referred to by the term rectangle. So C is the form usually referred to when a parallelogram is spoken of, without saying what kind of a parallelogram. C is also a rhomboid. (Why?) E is a square. (Why?) F is a rhombus. (Why?) This page is a rectangle; so also are the common panes of glass.

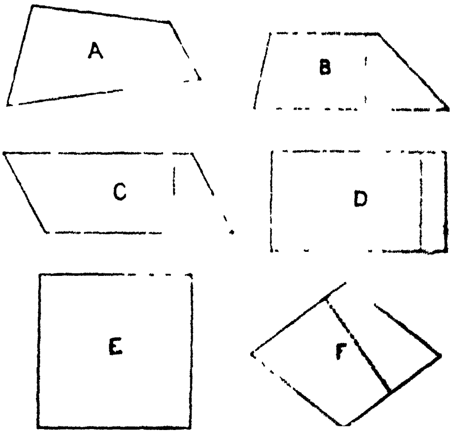


FIG.

100. ***A Diagonal*** is a line joining two angles of a figure, not adjacent.

Ill.—In common language, a diagonal is a line running "from corner to corner."

Ex. 1. To construct a square, having given a side; or, in other words, to construct a square on a given line.

---

* The pupil should be able to give this and all similar definitions at length. Thus, A Square is a surface inclosed by four equal right lines making right angles with each other.

1st Method.—Let A be the given side. Draw the indefinite line OX, and lay off OM = A. At M erect a perpendicular MY, as taught in (44). On this take MN = A. From N and O as centres, with a radius equal to A, describe arcs intersecting, as at P. Draw NP and PO.

2d Method.—Let Q be the given side. Construct equal angles at the extremities of Q, and produce the sides till they meet, and one of them till it will meet another side of the square proposed. With S as a centre, and ST or SR as radius, describe a semicircle. Draw RV, and it forms a right angle at R. The construction can now be finished as before.

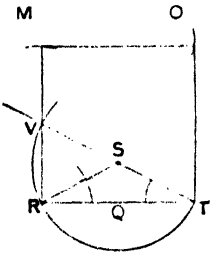


FIG. 80.

Ex. 2. Construct a rhombus whose side is 2 inches, and one of whose acute angles is ⅔ of a right angle.

Ex. 3. Construct a rectangle whose adjacent sides are 3 and 5.*

Ex. 4. Construct a rhomboid whose adjacent sides are 3 and 7, and their included angle ½ a right angle.

Ex. 5. How many diagonals has a triangle? How many has a quadrilateral? How many has a figure with five sides (a pentagon)? Of six? Of eight?

101. ***The Area*** of a surface is the number of times it contains some other surface taken as a unit of measure; or it is the ratio of one surface to another assumed as a standard of measure.

102. The *Unit of Area* usually assumed is a square, a side of which is some linear unit: thus, a *square inch*, a *square foot*, a *square yard*, a *square mile*, etc. By these terms is meant a square 1 inch on a side, one foot on a side, one yard on a side, etc.

The acre is an exception to the general rule of assuming the square on some linear unit as the unit of area, there being no linear unit in use whose length is the side of a square acre.

Ill.—The area of a board is the number of squares 1 foot on a side which it would take to cover it. The area of a floor may be spoken of in square yards, and is the same as the number of square yards of carpeting it would take to cover it.

---

* Take any convenient unit, as ½ inch, 1 inch.

103. *The Altitude* of a parallelogram is the distance between its opposite sides; of a trapezoid, it is the distance between its parallel sides; of a triangle, it is the distance from any vertex to the side opposite or to that side produced.

104. *The Bases* of a parallelogram or of a trapezoid are the sides between which the altitude is conceived as taken; of a triangle, it is the side to which the altitude is perpendicular.

Ill.—The dotted lines in B, C, D, and F, *Fig.* 79, represent altitudes. When the altitude is the distance between two parallels, the figure has two bases. The altitude of a parallelogram may be reckoned between either pair of parallel sides, but it is most common to conceive it as the distance between the two longer sides. The altitude of a rectangle is the same as either side to which it is parallel. A triangle may have three altitudes, and any side of a triangle may be conceived as its base. In *Fig.* 81, AB is conceived as the base in each case, and CD the altitude.

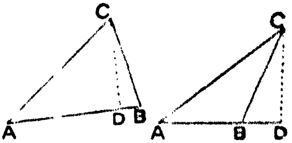


Fig. 81.

Ex. What side of a triangle must you conceive as the base, in order that the altitude shall fall upon it, and not upon its prolongation? From what angle will the altitude be reckoned in such a case?

105. *Theorem.*—*The area of a rectangle is the product of its two adjacent sides; or, what is the same thing, the product of its altitude and base.*

Ill.—Let ABCD represent a rectangle, of which AB is 8 units long, and AC 5. Now, let us conceive a square *a* constructed on one of these units. Using this surface as the unit of area, it is evident that in the rectangle *c*AB*d* there will be 8 such. Hence, the area of this rectangle is 8 (square units). Now, drawing parallels to the base through the several points of division of the altitude, it is evident that the whole rectangle ABCD is made up of as many rectangles like *c*AB*d* as there are units in the altitude—in this case 5. Hence the whole area is 5 times the area of *c*AB*d*, *i. e.*, 5 times 8 (square units) = 40 (square units).

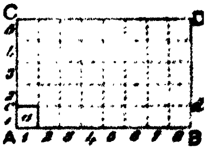


Fig. 82.

N.B.—*The pupil should be careful to observe that the language "product of base into altitude," is only a convenient form of abbreviated expression. It is* just as absurd to talk about multiplying a line by a line, as to talk about multiplying dollars by dollars. Thus 8 inches in length can be taken 5 times, and makes 40 inches in length. But what does 8 inches in length, multiplied by 5 inches *in length* mean? Or what is 8 dollars taken 5 *dollars* times? The multiplier must always be an abstract number, and the product be like the multiplicand, from the very nature of multiplication. With this the explanation given above agrees. When we say that the area of ABCD = 8 × 5, we mean 5 times 8 square units, which equals 40 square units.

106. Theorem.—*The area of any parallelogram is the same as the area of a rectangle having the same base and altitude as the parallelogram, and hence is the product of its base and altitude.*

ILL.—This truth is easily illustrated by cutting out a parallelogram, as ABCD. Then, cutting off the triangle DEC, being careful to make DE perpendicular to BC, and placing DC upon AB so as to bring the triangle DEC into the position AFB, the two parts will just make up the rectangle AFED. Hence we see that the area of ABCD is the same as the area of AFED, which latter is a rectangle having the same base AD, and the same altitude ED, as the given parallelogram.

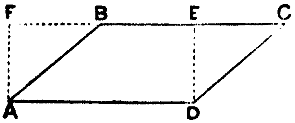


FIG. 83.

107. Theorem.—*The area of a triangle is half the product of its base and altitude.*

ILL.—To illustrate this truth, cut out two triangles A and B *just alike*. By placing them together, a parallelogram can be formed whose base and altitude are the same as the triangle. The area of the parallelogram is the product of its base and altitude. Hence the area of one of the triangles is one-half the product of its base and altitude.

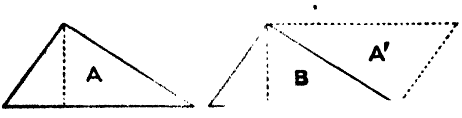


FIG. 84.

In fact, by cutting one of the triangles, as A, into two triangles, its parts can be put with B so as to make a *rectangle* having the same base and altitude as the triangles. [The pupil should do it.]

108. Theorem.—*The area of a trapezoid is the product of its altitude into the line joining the middle points of its inclined sides.*

ILL.—To illustrate this truth, cut out any trapezoid, as ABCD, and through the middle of the inclined sides, as *a* and *b*, cut off the triangles **A***am* and **B***bn*, being careful to cut in lines *am* and *bn* perpendicular to the base. These can be applied as indicated in the figure, so as to fill out the rectangle *omnp*. Hence we see that the area of the trapezoid is just equal to the product of its altitude into the line joining the middle points of its inclined sides, as *ab*.

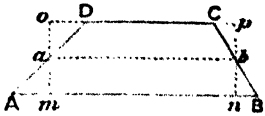


Fig. 85.

Ex. 1. How many square yards of plastering in the walls of a room 20 feet by 30, and 15 feet high, including the ceiling?

Ans. 233⅓.

Ex. 2. A salesman is selling a piece of velvet which is worth $8 per yard. The velvet is cut "on the bias," as the technical phrase is, *i.e.*, obliquely, instead of square across. The piece he is selling is measured along the selvedge in the usual way half a yard. He is disposed to charge the customer somewhat more than $4. Is he right? The customer claims that he is getting but half a yard of velvet, and so ought to pay but $4. Is he right?

Ans. Both are right,—the salesman in his demand, and the customer in his statement. How is it?

Ex. 3. There are two parallel roads one mile apart. **A** has a farm which extends along one of the roads half a mile, and the lines run perpendicularly from one road to the other. **B** has a farm lying between the same roads, and half a mile front on each road, but running obliquely across. Which has the larger farm?

Ex. 4. Of the four triangles **ACB**, **ADB**, **AEB**, and **AFB**, *Fig.* 86, which has the greatest area, **CF** being parallel to **AB**?

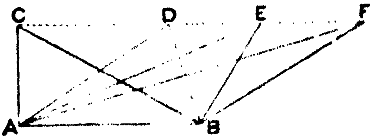


Fig. 86.

Ex. 5. Which is the largest triangle which can be inscribed in a semicircle, having the diameter for its base?

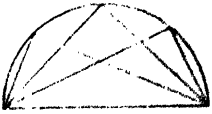


Fig. 87.

Ex. 6. Can you vary the area of a triangle while the sides remain of the same length? Can you vary the area of a quadrilateral while the sides remain of the same length?

Ex. 7. If you have two lines each 5 inches long, and two each 3 inches long; into what kind of a parallelogram must you form them in order to have its area the greatest?

Ex. 8. Rough boards are usually narrower at one end than at the other, for which reason the lumberman measures their width in the middle. What is the number of square feet in the following:

12 boards 16 feet long, 10 inches wide (in the middle);
15 boards 11 feet long, 9 inches wide " " ;
8 boards 10 feet long, 13 inches wide " " ?

What principle is involved in such measurement?

Ex. 9. What is the area of a triangle whose altitude is 6 feet, and base 10 feet? Are these elements sufficient to fix the *form* of the triangle?

Ex. 10. If a line be drawn from any angle of a triangle to the middle of the opposite side, what is the relation of the areas of the two partial triangles? Why?

THE PYTHAGOREAN PROPOSITION.

109. Theorem.—*The square described on the hypotenuse of a right angled triangle is equivalent to the sum of the two squares described on the other two sides.*

Ill.—The meaning of this proposition may be illustrated thus: Let ABC be a right angled triangle, right angled at C, and the sides AC and CB be 4 and 3 respectively. Then measuring AB, it will be found to be 5, and we observe that $4^2 + 3^2 = 5^2$. This is also seen from the figure, in which the square on AC contains $4^2 = 16$ square units, and that on CB $3^2 = 9$; while that on AB contains $5^2 = 25$, *i. e.*, as many as on both the other sides. We cannot so readily *illustrate* the truth of the proposition when the ratio of the sides is any other than that of 3, 4, and 5, but it is equally true in all cases, as will be proved in the next part of this book.

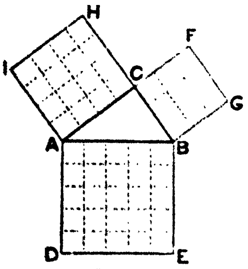


Fig. 83.

Ex. 1. Can you make a right angled triangle whose sides shall be 5, 8, and 10?

Sug.—As 10 is the longest side, it will have to be the hypotenuse. Now $5^2 + 8^2 = 25 + 64 = 89$. But $10^2 = 100$. Hence, 10 is too long for the hypotenuse of a right angled triangle whose other sides are 5 and 8.

Ex. 2. Can you make a right angled triangle whose sides shall be 9, 12, and 15?

Ex. 3. A carpenter has framed the four sills of a building together, and placed them on the foundation. He then wishes to adjust them so that the angles shall be right angles. He places one end of his *ten foot pole ab at a, 6 feet from c;* and, holding it in position, orders his attendants to move the sill AB to the right. How far will the end *b* of the pole be from *c* when the angle B is a right angle?

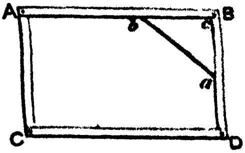


Fig. 89.

Ex. 4. A gate is to be 10 feet long and 4 feet high. How long must the brace be to go in as a diagonal and hold the gate in the form of a rectangle?

Ex. 5. The angles of a room are all right angles, and its dimensions are 20 feet by 30 on the floor, and 15 feet high. What is the length of the longest diagonal extending from one corner on the floor to the opposite corner in the ceiling?

Ans. A little more than 39 feet.

Ex. 6. The numbers 3, 4, and 5 are much used by artizans as parts of a right angled triangle. Will any equi-multiples of them answer the same purpose, as twice them, *i. e.*, 6, 8, and 10; or three times them, as 9, 12, and 15, etc.?

Ex. 7. In an obtuse angled triangle, is the square of the side opposite the obtuse angle greater or less than the sum of the squares of the other two sides? How is it with the square of the side opposite an acute angle?

Sug.—In the right angled triangle ABC, $\overline{AC}^2 = \overline{CB}^2 + \overline{AB}^2$. In the obtuse angled triangle C'B is equal to CB in the right angled triangle. But $\overline{AC'}^2$ is greater than $\overline{AC}^2$; hence $\overline{AC'}^2 > \overline{BC'}^2 + \overline{AB}^2$. By a similar inspection the other case may be determined.

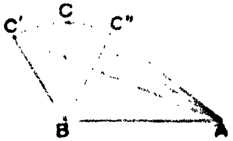


Fig. 90.

✗ ***110. Prob.***—*To find a mean proportional between two lines.*

Solution.—I wish to find a mean proportional between the lines M and N, *i. e.*, a line x, such that

$$M : x :: x : N, \text{ whence } x^2 = M \times$$
$$x = \sqrt{M \times N}.$$

I draw a line AB equal to the sum of M and N, making DB = M, and AD = N. I draw a

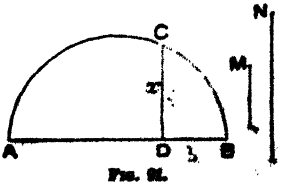


Fig. 91.

semicircumference on AB, and at D erect CD perpendicular to AB. CD is x, the mean proportional required.

Ex. 1. To construct a square which shall be equal in area to a given rectangle.

Sug.—Draw any rectangle. Then find a mean proportional between its adjacent sides as described above. A square constructed on this line will be equal in area to the rectangle; since, if x is the side of the square, and M and N are the adjacent sides of the rectangle, $x^2 = M \times N$. But x^2 is the area of the square, and $M \times N$ is the area of the rectangle.

Ex. 2. To find the square root of 15 by means of the ruler and compasses.

Sug.—Since $15 = 3 \times 5$, if $DB = 3$ and $AD = 5$, *Fig.* 91, x (CD) $= \sqrt{3 \times 5} = \sqrt{15}$. Therefore, making a figure having DB and AD of these lengths, CD can be measured, and thus the square root of 15 obtained, approximately, in numbers.

N. B.—*In such a case* CD *represents exactly the required root, although we may not be able to express the value exactly in numbers.* In this case geometry *does exactly what arithmetic can only do approximately.*

Ex. 3. Draw a line which shall represent, exactly, the square root of 5.

Sug.—Make $DB = 1$, and $AD = 5$.

Ex. 4. Draw a rectangle whose adjacent sides are 2 and 3, and then draw a square of the same area.

---

111. *Theorem.*—*The areas of similar triangles are to each other as the squares of their homologous sides.*

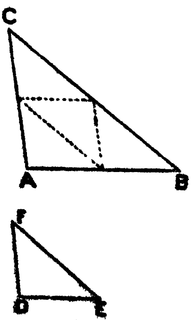


Fig. 92.

Ill.—The meaning of this is, that if ABC and DEF are similar, and any side of ABC is 2 times as great as the homologous side of DEF (as is the case in the figure, CB being = 2FE, CA to 2FD and AB to 2DE) the area of ABC is 4 times the area of DEF. In fact, in a simple case like this, we can divide ABC into four triangles exactly equal to DEF, as is done by the dotted lines.

Ex. 1. A and B have triangular pieces of land, which are similar to each other, and similarly situated. But A's front is to B's as 5 to 3; how much more land has A than B?

Ans. $2\frac{7}{9}$ times as much.

Ex. 2. In order that one triangle may be similar to and 4 times as great as another, how must any side of the first compare with the homologous side of the second?

Ex. 3. In order that the areas of two similar triangles may be to each other as 4 to 9, what must be the ratio of their homologous sides?

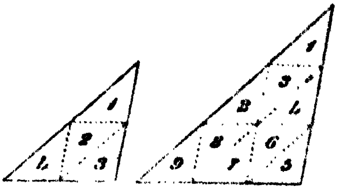


Fig. 93.

---

112. Theorem.—*The homologous sides of similar triangles are to each other as the square roots of their areas.*

This theorem is involved in the theorem that the areas of similar triangles are to each other as the squares of their homologous sides. It is illustrated in the preceding examples.

Ex. Construct a triangle with one of its sides 2 in length. Then construct a similar triangle 1½ times as large. What must be the length of the side of the second triangle which is homologous with the side 2 of the first.

Solution.—Let CAB be the given triangle, whose side AB is 2. Since the second is to be 1½ times as great as the first, the ratio of the areas is 2 : 3. Hence, $\sqrt{2} : \sqrt{3} :: AB$ (or 2) : x, the side of the required triangle homologous with side 2 of the given triangle. Construct the square roots of 2 and 3, as *ab* and *ac* in the figure, and then find a fourth proportional to *ab*, *ac*, and AB. This is found to be *ay*. Taking DE = *ay*, construct on it a triangle DEF similar to ABC, and it will be 1½ times as large.

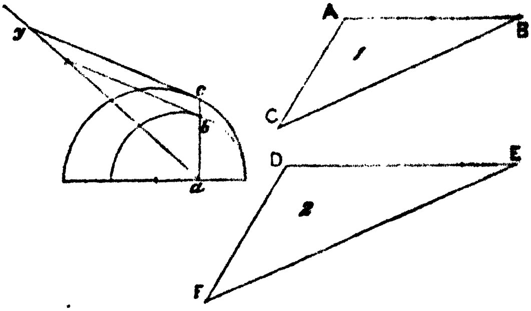


Fig. 94.

THE AREA OF A CIRCLE.

113. Theorem.—*The area of a circle whose radius is r, is πr^2, i.e., 3.1416 times the square of its radius.*

Ill.—If we take a circle whose radius is r and circumscribe about it a square ABCD, we observe that the area of this square is $4r^2$. Hence we see that the area of a circle is *less* than 4 times the square of its radius. Again, drawing two diameters EF and GH at right angles to each other, and joining their extremities, we have the inscribed square GEHF. The area of this square is equal to the area of the two triangles GEF and EHF. But area $GEF = \frac{1}{2}GO \times EF = \frac{1}{2}r \times 2r = r^2$; and in like manner $EHF = r^2$. Hence area $GEHF = 2r^2$. We thus see that the area of a circle is *more* than two times the square of the radius. The area *of a circle is therefore somewhere between two and four times the square of its radius.* Just how many times r^2 the area is, we do not propose to find in this place, but only say that it has been found to be 3.1416 times r^2. We must also remark that this is not *exact*; but it is near enough for practical purposes. In fact, nobody knows exactly how many times the square of the radius the area of a circle is.

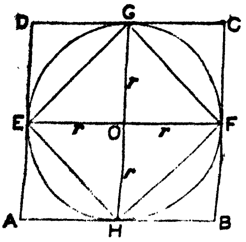


Fig. 95.

Ex. 1. If you cut from a square the largest possible circle, show that you cut away a little less than $\frac{1}{4}$ of the square, or more exactly .2146 of it.

Ex. 2. What is the area in acres of a circle whose diameter is 3 miles? *Ans.* 4523.904.

Ex. 3. A horse is so tied to a tree that he can graze on every side of it to a distance of 100 feet. What is the area in acres over which he can graze? *Ans.* A little less than $\frac{3}{4}$ of an acre.

Ex. 4. What is the area of a circle whose radius is 1?

[Remember this result.]

Ex. 5. What is the area of a circle whose radius is 2? 3? 4? How do these areas compare with the area of a circle whose radius is 1?

---

114. Theorem.—*The areas of circles are to each other as the squares of their radii.*

Ill.—This is readily seen from the last theorem. Thus the area of a circle whose radius is 5 is 25π; and of one whose radius is 6, the area is 36π. Now, the ratio of these areas $25\pi : 36\pi$ is the same as $25 : 36$, *i. e.*, as the squares of the radii of the two circles.

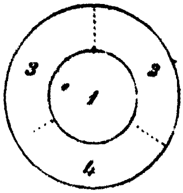


FIG. 96.

Ex. 1. In the figure the radius of the outer circle is twice that of the inner. How do their areas compare? How do the 4 parts into which the larger circle is divided compare with each other?

Ex. 2. The radii of 2 circles are 3 and 5 respectively; what is the relation of their areas?

Ans. $9 : 25$; or one is $2\frac{7}{9}$ times as large as the other.

Ex. 3. I have a circle whose radius is 5, and wish to make another whose area is twice as great; what must be its radius?

Ans. $\sqrt{50}$, or 7.071 nearly.

Ex. 4. Can we compare the areas of circles by means of the squares of their diameters as well as by means of the squares of their radii? How much greater is the square of the diameter of any circle than the square of the radius?

Ex. 5. Two 5-inch stovepipes run together into one 7-inch pipe. Is the capacity of the one pipe equal to that of the two?

Ex. 6. Two men bought grindstones of equal thickness. The stones cost $4 and $9 respectively. One was 2 feet in diameter and the other 3. What was the difference in the rates paid?

SECTION VIII.

OF POLYGONS.

115. A Polygon is a portion of a plane bounded by straight lines.

The word polygon means many-angled; so that with strict propriety we might limit the definition to plane figures with five or more sides. This limitation in the use of the word is frequently made.

116. A polygon of three sides is a *triangle*; of four, a *quadrilateral*; of five, a *pentagon*; of six, a *hexagon*; of seven, a *heptagon*; of eight, an *octagon*; of nine, a *nonagon*; of ten, a *decagon*; of twelve, a *dodecagon*.

117. *A Regular Polygon* is a polygon whose sides are equal each to each, and whose angles are equal each to each.

118. *The Perimeter* of a polygon is the distance around it, or the sum of the bounding lines.

---

119. *Theorem.*—*Any polygon may be divided by diagonals drawn from any angle, into as many triangles as the polygon has sides, less two sides.*

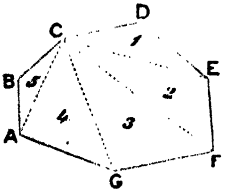


FIG. 97.

ILL.—In the figure the polygon has 7 sides. By drawing the diagonals from C to the other angles, we divide the polygon into 5 (7—2) triangles.

---

120. *Theorem.*—*The sum of the angles of any polygon is twice as many right angles as the polygon has angles (or sides), less four right angles.*

ILL.—Draw a polygon, as ABCDEFG, and the arcs *a*, *b*, *c*, *d*, *e*, *f*, *g*, measuring its angles. *With the same radius* draw a circle. Beginning at some point, as O, lay off $OA = a$, $AB = b$, $BC = c$, $CD = d$, $DE = e$, $EF = f$, and $FG = g$. It is found in this case that the sum of these measures is two circumferences and a half. Now, one circumference is the measure of 4 right angles. Hence, 2½ circumferences measure 2½ × 4 = 10 right angles. Thus it appears that the sum of all the angles of the polygon is 10 right angles. This agrees with the theorem; for, by that, the sum should be 2 *right angles* × 7 — 4 right angles, which is 10 right angles.

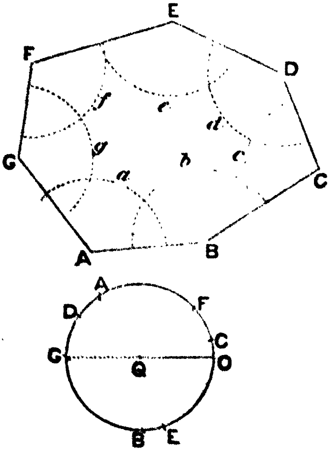


FIG. 98.

---

121. *Prob.*—*To draw a regular polygon.*

SOLUTION.—Draw a circle, and divide the circumference into as many equal arcs as the polygon has sides. The chords of these arcs will constitute the perimeter of the polygon.

The practical difficulty lies in dividing the circumference as required. The circumference can be divided into 6 equal arcs by (*55*). Drawing radii to these points of division, and bisecting the included angle, a division into 12 equal parts is effected. These can be again bisected, and the division into 24 equal parts effected, etc. Again, the circumference can be divided into 4 equal parts by drawing two diameters at right angles to each other (see *Fig.* 95). These arcs can be bisected as indicated above, and the division into 8 equal parts effected. Bisecting the latter arcs, we have 16 equal parts, etc. There is also a way to divide the circumference into 10 equal parts, but it is too difficult to be given here. For all regular polygons except those of 3, 6, 12, 24, etc., and 4, 8, 16, etc., sides, the pupil, at this stage of his progress, is expected to effect the division *by trial.*

EXERCISES.

1. By drawing diagonals from any one angle, into how many triangles can a pentagon* be divided? Show it with a figure. Into how many an octagon? A dodecagon? A nonagon? A hexagon?

2. What is the sum of the angles of a hexagon? Determine the number mentally, and then measure the angles geometrically, as in the solution of (*120*), observing that the latter result verifies the former. In like manner determine the sum of the angles of a pentagon. Of an octagon. Of a decagon. Of a nonagon. Of a triangle. Of a quadrilateral.

3. If the angles of a hexagon are equal each to each—that is, if the hexagon is equiangular—what is the value of any one angle?

Ans. 1⅓ right angles.

[NOTE.—A regular polygon is equiangular.]

4. What is the value of any angle of a regular octagon? Of a regular pentagon? Of a regular dodecagon?

Answer to the last, 1⅔ right angles.

5. Construct a regular dodecagon.

6. Construct a regular heptagon.

SUG'S.—Observing that as the chord for the *hexagon* is the radius, and hence the chord for the heptagon is *a little less*, we can readily find *by trial* just how wide to open the dividers so that they shall step around the circumference at 7 steps. This is not a very scientific way of constructing a figure, it is true, but it is the only way we can get the chord in this case.

FIG. 99.

---

* Polygons are not to be assumed regular unless they are so designated.

7. Construct a regular octagon.

SUG.—See the general solution (*121*).

8. Construct a regular nonagon.

SOLUTION.—*First get a quarter of the circumference by marking the points where two diameters at right angles to each other would cut the circumference.* AX is an arc of 90°. Then from A take AY = 60° by using radius as a chord. YX is therefore an arc of 30°. Divide this into three equal parts *by trial.* Measure YB equal to two-thirds of YX, and AB and BC are arcs of 40°, and the chords AB and BC are chords of the regular nonagon.

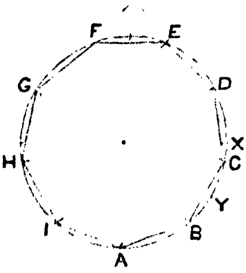


FIG. 100.

9. To draw a five-point star.

SOLUTION.—Draw a circle, and dividing the circumference into five equal parts, join the alternate points of division, as in the figure.

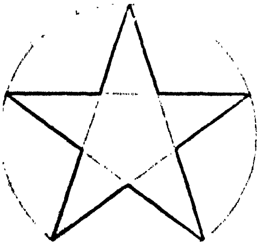


FIG. 101.

10. To circumscribe a square about a circle (*56*). Also an equilateral triangle, and a regular hexagon.

SYNOPSIS OF PLANE FIGURES.

- Plane Figures.
  - What?
  - Classes of.
    - Polygons.
      - What? Sides. Perimeter Diagonal.
      - Triangles.
        - What? Altitude. Base.
        - Classified by sides.
          - Scalene.
          - Isosceles.
          - Equilateral.
        - Classified by angles.
          - Acute.
          - Right.
          - Obtuse.
      - Quadrilaterals.
        - What?
        - Trapezium.
        - Trapezoid.
        - Parallelogram.
          - Rhombus.
          - Rhomboid.
          - Rectangular.
            - With unequal sides.
            - Square.
      - Pentagon. Hexagon. Heptagon. Octagon. Nonagon, etc.
        - Regular. What?
    - Bounded by Curves.
      - Circle.
        - What?
        - Circumference.
        - Centre.
        - Radius, Diameter.
      - Conic Sections.*
        - Ellipse.
        - Parabola.
        - Hyperbola.
      - Higher Plane Curves.*

---

* These are inserted simply to give completeness. Of course, the student is not expected to know more than their names.

PART II.

THE FUNDAMENTAL PROPOSITIONS OF ELEMENTARY GEOMETRY, DEMONSTRATED, ILLUSTRATED, AND APPLIED.

CHAPTER I.

PLANE GEOMETRY.

SECTION I.

OF PERPENDICULAR STRAIGHT LINES.

PROPOSITION I.

182. Theorem.—*At any point in a straight line, one perpendicular can be erected to the line, and only one, which shall lie on the same side of the line.*

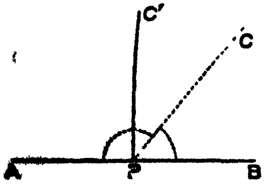


Fig. 102.

Dem.—Let AB* represent any line, and P be any point therein; then, on the same side of AB there can be one and only one perpendicular erected at P. For from P draw any oblique line, as PC, forming with AB the two angles CPB and CPA. Now, while the extremity P, of PC, remains at P, conceive the line PC to revolve so as to increase the less of the two angles, as CPB, and decrease the greater, as CPA. It is evident that for a certain position of CP, as C'P, these

---

[*] In class recitation the pupil should go to the blackboard, after having had his proposition assigned him, and first draw the figure required for the demonstration. This should be done neatly, accurately, with dispatch, and without any aid. The figure being complete, he stands at the board, pointer in hand, enunciates the proposition, and then gives the demonstration as it is in the text, pointing to the several parts of the figure as they are referred to.

angles will become equal. In this position C′P becomes perpendicular to AB (*26*).* Again, if the line C′P revolve from the position in which the angles are equal, one angle will increase and the other diminish; hence there is *only one* position of the line on this side of AB in which the adjacent angles are equal. Therefore there can be one and only one perpendicular erected to AB at P, which shall lie on the same side of AB. Q. E. D.

123. COR. 1.—*On the other side of the line a second perpendicular, and only one, can be drawn from the same point in the line.*

124. COR. 2.—*If one straight line meets another so as to make the angle on one side of it a right angle, the angle on the other side is also a right angle, and the first line is perpendicular to the second.*

125. COR. 3.—*If two lines intersect so as to make one of the four angles formed a right angle, the other three are right angles, and the lines are mutually perpendicular to each other.*

DEM.—Thus, if CEB is a right angle, CEA, being equal to it, is also a right angle. Then, as AEC is a right angle, the adjacent angle AED is a right angle, since they are equal. Also, as CEB is a right angle, and BED equal to it, BED is a right angle. Hence CD being perpendicular to AB, AB is perpendicular to CD, as it meets CD so as to make the adjacent angles AEC and AED, or CEB and BED equal to each other (*43*).

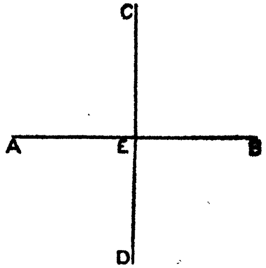


FIG. 103.

PROPOSITION II.

126. Theorem.—*When two straight lines intersect at right angles, if the portion of the plane of the lines on one side of either line be conceived as revolved on that line as an axis until it coincides with the portion of the plane on the other side, the parts of the second line will coincide.*

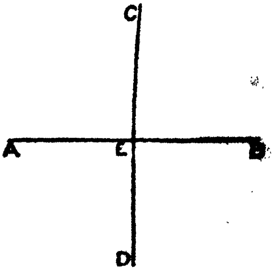


FIG. 104.

DEM.—Let the two lines AB and CD intersect at right angles at E; and let the portion of the plane of the lines on the side of CD on which B lies be conceived to revolve on the line CD as an axis,† until it falls in the portion of the plane on the other side of CD. Then will EB fall in and coincide with AE.

---

* *When a preceding principle is referred to, it should be accurately quoted by the pupil.*

† As if the paper, which may represent the plane of the lines, were folded in the line CD. It is important that this process be clearly conceived, as it is to be made the basis of many subsequent demonstrations.

For, the point E being in CD, does not change position in the revolution; and, as EB remains perpendicular to CD, it must coincide with EA after the revolution, or there would be two perpendiculars to CD on the same side and from the same point, E, which is impossible (*122*). Hence EB coincides with EA. Q. E. D.

PROPOSITION III.

127. Theorem.—*From any point without a straight line, one perpendicular can be let fall upon that line, and only one.*

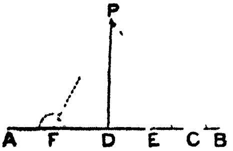


FIG. 105

DEM.—Let AB be any line, and P any point without the line; then one perpendicular, and only one, can be let fall from P upon AB

For, conceive any oblique line, as PC, drawn, making the angle PCB>PCA. Now, while the extremity P of this line remains fixed, conceive the line to revolve so as to make the greater angle PCB decrease, and the less angle PCA increase. At some position of the revolving line, as PD, the two angles which it makes with the line AB will become equal. When these adjacent angles are equal, the line, as PD, is perpendicular to AB (*26*, *43*). Moreover, there is *only one* position of the line in which these angles are equal; hence, only one perpendicular can be drawn from a given point to a given line. Q. E. D.

PROPOSITION IV.

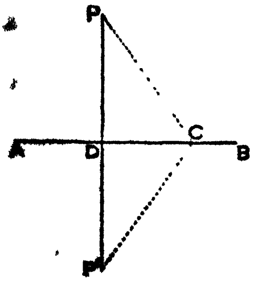


FIG. 106.

128. Theorem.—*From a point without a straight line, a perpendicular is the shortest distance to the line.*

DEM.—Let AB be any straight line, P any point without it, PD a perpendicular, and PC any oblique line; then is PD<PC.

Produce PD, making DP' = PD, and draw P'C. Then let the portion of the plane of the lines above AB be revolved upon AB as an axis, until it coincides with the portion below AB. Since PP' and AB intersect at right angles, PD will fall in DP' (126); and, since PD = DP', P will fall at P', and PC = P'C, since they coincide when applied. Finally, PP' being a straight line is shorter than

PCP′, which is a broken line, since a straight line is the shortest distance between two points. Hence PD, the half of PP′, is less than PC, the half of the broken line PCP′. Q. E. D.

PROPOSITION V.

***129. Theorem.**—If a perpendicular be erected at the middle point of a straight line,*

1st. *Any point in the perpendicular is equally distant from the extremities of the line.*

2d. *Any point without the perpendicular is nearer the extremity of the line on its own side of the perpendicular.*

Dem.—1st. Let PD be a perpendicular to AB at its middle point D. Then, O being any point in the perpendicular, OA = OB.

For, revolve the figure OBD upon OD as an axis until it falls in the plane on the other side of PD. Since ODB and ODA are right angles, DB will fall in DA (*126*); and, since DB = DA, B will fall at A. Hence, OA and OB coincide, and OA = OB.

2d. O′ being any point without the perpendicular on the same side as B, O′B < O′A.

For, drawing O′A and O′B, let O be the point at which O′A cuts the perpendicular. Draw OB. Now O′B < BO + OO′, since O′B is a straight and O′OB is a broken line. But, as OA = OB, we may substitute it in the inequality, and have O′B < OA + OO′, which sum = O′A.

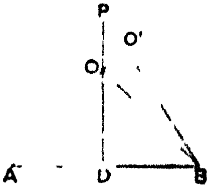


FIG. 107.

***130.** Cor.—If each of two points in one line is equally distant from the extremities of another line, the former line is perpendicular to the latter at its middle point.*

Dem.—Every point equally distant from the extremities of a straight line lies in a perpendicular to that line at its middle point, by the proposition. But, two points determine the position of a straight line. Hence, two points, each equally distant from the extremities of a straight line, determine the position of the perpendicular at the middle point of the line.

EXERCISES.

1. ***Prob.**—To erect a perpendicular to a given line at a given point in the line.*

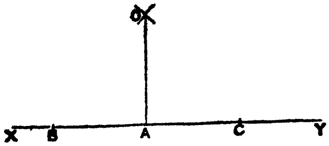


Fig. 108.

Solution.—[*The process* is given in (***44***), and should be repeated here exactly as given there, with the *reasons for the solution*, as follows.] A is one point in the line OA, which is equally distant from B and C, by construction, and O is another. Hence OA is perpendicular to BC at A, by (***130***).

2. ***Prob.***—*To bisect a given line.*

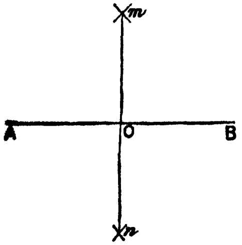


Fig. 109.

Solution.—[For the *process* see (***39***). The student should first do it as he did then. The reason why this process bisects AB is as follows.] Since *m* is one point equally distant from the extremities A and B, and *n* another, there are two points in *mn* each equally distant from the extremities of AB. Hence *mn* is perpendicular to AB at its middle point O, by (***130***). [The reason for the process in *Fig.* 20 is the same. Let the student give this method, and show how the corollary (***130***) applies.]

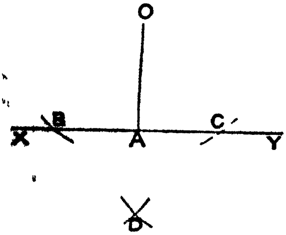


Fig. 110.

3. ***Prob.***—*From a point without a given line, to let fall a perpendicular upon the line.*

Solution.—[Repeat the process as in (***45***), and give the reason for it as follows.] O is one point equally distant from B and C, and D is another. Hence a line drawn from O to D is perpendicular to BC by (***130***).

4. Wishing to erect a line perpendicular to AB at its centre, I take a cord or chain somewhat longer than AB, and, fastening its ends at A and B, take hold of the middle of the cord or chain and carry it as far from AB as I can, first on one side and then on the other, sticking pins at the most remote points, as at P and P'. These points determine the perpendicular sought. What is the principle involved?

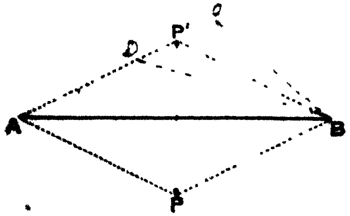


Fig. 111.

5. Two boys are skating together on the ice, and both start from

the same point at the same time, one skating directly to the shore and the other obliquely. They both reach the shore at the same time. Which skates the faster? What principle is involved?

6. Several persons start at different times from the same point in a straight road that runs along a wood, and each travels directly away from the road. Will they come out at the same, or at different points on the opposite side of the wood? What principle is involved? What is the geometrical language for the colloquial phrase "Directly away from the road"?

7. If I go from A to B, *Fig.* 111, by first passing over AP, will I gain or lose in distance by going on a little farther in the direction of AP before I turn and go straight to B? What principle is involved? Would I gain or lose by stopping short of P on the line AP? Why?

SECTION II.

OF OBLIQUE STRAIGHT LINES.

PROPOSITION I.

131. Theorem.—*When an oblique line meets another straight line forming two adjacent angles, the sum of these angles is two right angles.*

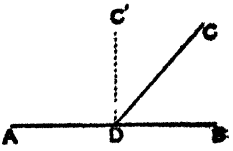


FIG. 112.

DEM.—Let the oblique line CD meet the straight line AB forming the two adjacent angles CDB and CDA; then CDB + CDA equals two right angles

For suppose CD to revolve toward the position of the perpendicular C'D; the angle CDB will increase at the same rate that CDA diminishes; hence their sum will remain constant (*i. e.*, the same). But, when CD becomes perpendicular, the sum of the adjacent angles formed with AB is two right angles by definitions (26, 43). Therefore CDB + CDA = two right angles. Q. E. D.

132. COR.—*The sum of all the consecutive angles formed by any number of lines meeting a given line on the same side and at a given point is two right angles.*

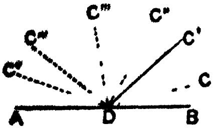


Fig 113

DEM.—Thus $ADC^v + C^vDC^{iv} + C^{iv}DC''' + C'''DC'' + C''DC' + C'DC + CDB = ADC' + C'DB$, which sum is two right angles by the proposition. Or, in general terms, the angles thus formed can always be united into two groups, constituting respectively the two adjacent angles formed by one line meeting another.

133. DEF.—Two angles whose sum is two right angles, are called *Supplemental Angles*. Hence, the *Supplement* of an angle is what remains after subtracting it from two right angles.

PROPOSITION II.

134. Theorem.—*When any two straight lines intersect, the opposite or vertical angles are equal to each other, and the sum of the four angles formed is four right angles.*

DEM.—Let AB and CE intersect at D; then CDA = the opposite angle BDE, ADE = the opposite or vertical angle CDB, and ADC + CDB + BDE + EDA = four right angles For, since CD meets AB, ADC is the supplement of CDB. (*131*, *133*.) Also, since BD meets CE, BDE is the supplement of CDB Hence ADC = BDE. In a similar manner ADE can be proved equal to CDB. [The student should give the proof.]

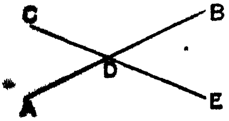


Fig. 114

Again, since ADC + CDB = two right angles, and BDE + EDA = two right angles, by adding the corresponding members together, we have ADC + CDB + BDE + EDA = four right angles.

135. COR.—*The sum of all the consecutive angles formed by any number of lines meeting at a common point is four right angles.*

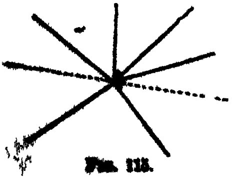


Fig. 115.

DEM.—The truth of this corollary is rendered apparent by drawing a line through the common vertex, and observing that the sum of all the angles on each side thereof is two right angles; whence the sum of all the angles on both sides, which is the same as the sum of all the consecutive angles formed by the line, is four right angles. [Let the student put letters on the figure, and demonstrate by means of it.]

PROPOSITION III.

136. Theorem.—*If two supplemental angles are so situated as to be adjacent to each other, the two sides not common will fall in the same straight line.*

DEM.—Let the sum of the two angles BOA and CO′D be two right angles. Prolong CO′, forming the angle DO′E. Then is DO′E supplemental to CO′D (*131*, *133*), and hence equal to BOA, which is supplemental to CO′D by hypothesis. Now, if AOB be placed adjacent to CO′D, the vertex O being at O′, and the side OA falling in O′D, OB will fall in O′E, since BOA = DO′E. Hence, when the angles are so situated, OB becomes the prolongation of CO′. Q. E. D.

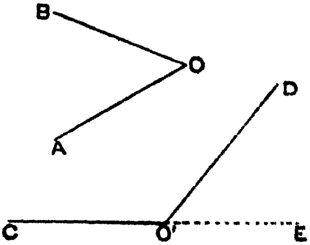


FIG 116.

PROPOSITION IV.

137. Theorem.—*If from a point without a straight line a perpendicular be drawn, oblique lines from the same point cutting the line at equal distances from the foot of the perpendicular are equal to each other; the angles which they form with the perpendicular are equal to each other; and the angles which they form with the line are equal to each other.*

DEM.—Let AB be any straight line, P any point without it, PD a perpendicular, and PC and PE oblique lines cutting AB at C and E, so that DC = DE; then PC = PE, angle CPD = angle DPE, and angle PCD = angle PED.

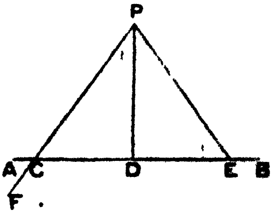


FIG. 117.

Revolve the figure PDE upon PD as an axis, until it falls in the plane on the other side of PD. Since AB is perpendicular to PD, DB will fall in DA; and, since DE = DC, E will fall at C. Now, as P remains stationary, the triangles PDE and PDC coincide. Hence, PC = PE, angle CPD = angle DPE, and angle PCD = angle PED. Q. E. D.

QUERY.—How does the equality of PE and PC follow from (*129*)?

PROPOSITION V.

138. Theorem.—*If from a point without a line a perpendicular be drawn to the line, and also from the same point two oblique* *lines making equal angles with the perpendicular, the oblique lines are equal to each other, cut the line at equal distances from the foot of the perpendicular, and make equal angles with it.**

DEM.—PD being a perpendicular to AB, and angle CPD equal to angle DPE, PC equals PE, CD equals DE, and PCD equals PED.

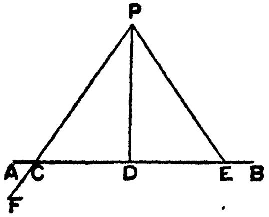


FIG 118.

Revolve the figure PDE upon PD as an axis, till it falls in the plane of PDC. Since angle EPD = angle CPD, PE will take the direction PC, and E will fall somewhere in the indefinite line PF. But, since PDE and PDC are right angles, DE will fall in DA (*126*) and E will fall somewhere in the indefinite line DA. Now, as E falls at the same time in PF and DA, it must fall at their intersection C. Hence, PE coincides with PC, and DE with DC. Therefore $PE = PC$, $DE = DC$, and angle $PED = PCD$. Q. E. D.

PROPOSITION VI.

139. Theorem.—*If from a point without a line a perpendicular be let fall on the line, and two oblique lines be drawn, the oblique line which cuts off the greater distance from the foot of the perpendicular is the greater.*

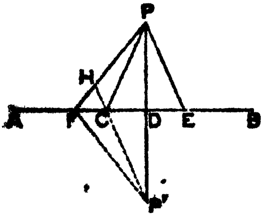


FIG. 119.

DEM.—Let AB be any straight line, P any point without it, and PC and PF two oblique lines of which PF cuts off the greater distance from the foot of the perpendicular; that is $DF > DC$. Then is $PF > PC$.

Produce PD, making $DP' = PD$, and draw P'F and P'C, producing the latter until it meets PF in H. Revolve the figure FPD upon AB as an axis, until it falls in the plane on the opposite side of AB. Since PP' is perpendicular to AB, PD will fall in DP'; and, since $PD = DP'$, P will fall at P'. Then $P'C = PC$ and $P'F = PF$. Now the broken line $PCP' <$ than the broken line PHP', since the straight line $PC <$ the broken line PHC. For a like reason the broken line $PHP' < PFP'$, since $HP' < HFP'$. Hence $PCP' < PFP'$, and PC the half of $PCP' < PF$ the half of PFP'. Q. E. D.

SCH.—If the two oblique lines to be compared lie on different sides of the perpendicular, as PF and PE, DF being greater than DE, lay off $DC = DE$, and draw PC. Then since $PC = PE$, if it is found less than PF, as in the demonstration, PE is less than PF.

---

* This proposition is the converse of the last. The significance of this statement will be more fully developed hereafter ([illegible]).

140. COR. 1.—*From a given point without a line, there can not be two equal oblique lines drawn to the line on the same side of a perpendicular from the point to the line.*

141. COR. 2.—*Two equal oblique lines drawn from the same point in a perpendicular to a given line, cut off equal distances on that line from the foot of the perpendicular.*

DEM.—For, if the distances cut off were unequal, the lines would be unequal.

EXERCISES.

1. Having an angle given, how can you construct its supplement? Draw any angle on the blackboard, and then construct its supplement.

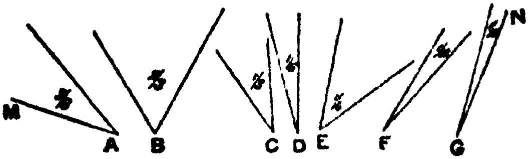


Fig 130

2. The several angles in the figure are such parts of a right angle as are indicated by the fractions placed in them. If these angles are added together by bringing the vertices together and causing the adjacent sides of the angles to coincide, how will MA and GN lie? Construct seven consecutive angles of these several magnitudes. How do the two sides not common lie? Why?

3. If two times A, B, two times D, three times E, three times C, three times G, two times F, in the last figure, are added in order, how will AM and GN lie with reference to each other? Why?

Ans. They will coincide.

4. If you place the vertices of any two equal angles together so that two of the sides shall extend in opposite directions and form one and the same straight line, the other two sides lying on opposite sides thereof, how will the latter sides lie? By what principle?

5. Upon what principle in this section may the common method of erecting a perpendicular at the middle of a straight line (*39*, *44*) be explained? Upon what the method of letting fall a perpendicular upon a straight line from a point without (*44*)?

6. A and B start at the same time, from the same point in a certain road; A travels directly to a point in another road at right angles to the first, and at ten miles from their intersection, and B travels directly toward a second point in the second road, which point is seven miles from the intersection. Both reach their destination at the same time. Which travels the faster? What principle is involved?

SECTION III.

OF PARALLELS.

PROPOSITION I.

142. Theorem.—*Two straight lines lying in the same plane and perpendicular to a third line are parallel to each other.*

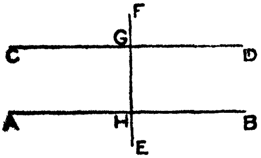


FIG. 121

DEM.—Let AB and CD be two straight lines lying in the same plane and each perpendicular to FE; then are they parallel.

For if AB and CD are not parallel, they will meet at some point if sufficiently produced (*66*). But, if they could meet, we should have two straight lines from one point (their point of meeting), perpendicular to the same straight line, which is impossible (*127*). Therefore, as the lines lie in the same plane and cannot meet how far soever they be produced, they are parallel. Q. E. D.

143. COR. 1.—*Through the same point one parallel can always be drawn to a given line, and only one.*

DEM.—Let AB be the given line, and G the given point, there can be one and only one perpendicular through G to AB (*127*.) Let this be FE. Now through G one and only one perpendicular can be drawn to FE. Let this be CD. Then is CD parallel to AB by the proposition. That there is only one such parallel, we shall assume as axiomatic.*

144. COR. 2.—*If a straight line is perpendicular to one of two parallels, it is perpendicular to the other also.*

DEM.—If FE is perpendicular to AB it is perpendicular to CD. For, if through G where FE intersects CD, a perpendicular be drawn to FE, it is parallel to AB by the proposition. But, by *Cor.* 1, there can be but one line through C parallel to AB. Hence the perpendicular to FE at C coincides with, or is, the parallel CD.

---

* Nous regarderons cette proposition comme évidente. F.-J. COMPAGNIE. See also CHAUVENET.

X

PROPOSITION II.

145. Theorem.—*Two straight lines which are parallel to a third, are parallel to each other.*

DEM.—Let AB and CD be each parallel to EF; then are they parallel to each other.

For draw HI perpendicular to EF; then will it be perpendicular to CD because CD is parallel to EF. For a like reason HI is perpendicular to AB Hence CD and AB are both perpendicular to HI, and consequently parallel. Q. E. D.

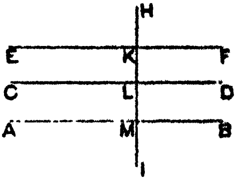


Fig 122

146. DEFINITIONS.—When two lines are cut by a third line the angles formed are named as follows:

Exterior Angles are those without the two lines, as 1, 2, 7, and 8.

Interior Angles are those within the two lines, as 3, 4, 5, and 6.

Alternate Exterior Angles are those without the two lines and on different sides of the secant line, but not adjacent, as 2 and 7, 1 and 8.

Alternate Interior Angles are those within the two lines and on different sides of the secant line but not adjacent, as 3 and 6, 4 and 5.

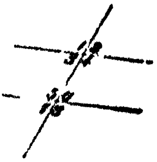


Fig. 123

Corresponding Angles are one without and one within the two lines, and on the same side of the secant line but not adjacent, as 2 and 6, 4 and 8, 1 and 5, 3 and 7.

PROPOSITION III.

147. Theorem.—*If two lines are cut by a third line, making the sum of the interior angles on the same side of the secant line equal to two right angles, the two lines are parallel.*

DEM.—Let AB and CD be met by the line EF, making ECD + FKB = two right angles; then are AB and CD parallel.

PROPOSITION IV.

150. Theorem.—*If two parallel lines are cut by a third line, the sum of the interior angles on the same side of the secant line is equal to two right angles.*

Dem.—Let the parallels AB and CD be cut by EF, then is DGK + GKB = two right angles.

For, if DGK is not the supplement of GKB, let LM be drawn through G so as to make MGK that supplement. Then, by the preceding proposition, LM is parallel to AB, and we have two parallels to AB through the point G, which is impossible (*143*). Hence, as no line but a parallel can make this interior angle the supplement of the other, the parallel makes it so. Q. E. D.

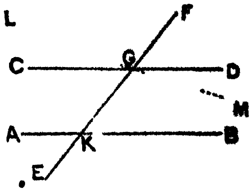


Fig. 127

[Let the student demonstrate this proposition as the preceding was demonstrated. In this case CD and AB are parallel by hypothesis, and HI being drawn perpendicular to one is perpendicular to the other also. When K falls at G, KI falls on CG, since from a point without a line only one perpendicular can be drawn to that line.]

151. Cor. 1.—*If two parallel lines are cut by a third line, the sum of either two exterior angles on the same side of the secant line is equal to two right angles.*

Dem.—FGD + EKB = two right angles. For FGD and GKB are both supplements of DGK (*133, 150*), and therefore equal to each other. For like reasons, EKB = KGD. Therefore, FGD + EKB = GKB + DGK = two right angles, by the proposition.

152. Cor. 2.—*If two parallel lines are cut by a third line, either two alternate interior, or either two alternate exterior, or either two corresponding angles, are equal to each other.*

Dem.—If CD and AB are parallel, CGK = GKB. For each is the supplement of KGD, the former by (*131*), the latter by (*150*). [Let the student show in like manner that AKG = KGD, FGD = AKE, CGF = EKB, FGD = GKB, and CGF = AKG.]

153. Cor. 3.—*Of the eight angles formed when one line cuts two parallels, the four acute angles are equal each to each, and the four obtuse angles; or, in case any one angle is a right angle, all the others are right angles.*

154. Sch.—The last two propositions and their corollaries are the *converse* of each other; *i. e.*, the hypotheses or data and the conclusions or things proved are exchanged. Thus, in Prop. III., the hypothesis is, that *The sum of the two interior angles on the same side of the secant line is equal to two right angles;* and the conclusion is, that *The two lines are parallel.* Now, in Prop. IV., the hypothesis is, that *The two lines are parallel;* and the conclusion is, that *The sum of the two interior angles on the same side of the secant line is two right angles.** [A clear conception of this scholium will save the student from confounding these propositions.]

PROPOSITION V.

155. *Theorem.*—*If two straight lines are cut by a third line making the sum of the interior angles on one side of the secant line less than two right angles, the two lines will meet on this side of the secant line, if sufficiently produced.*

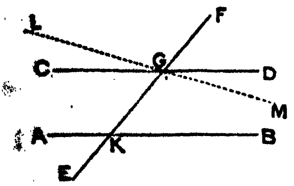


Fig. 128.

Dem.—Let AB and LM be cut by EF making MGK + FKB < two right angles; then will AB and LM meet on the side of EF on which MGK and FKB lie, if sufficiently produced.

For the angle which a parallel to AB through G makes with EF is the supplement of FKB. But by hypothesis MGK is less than this supplement. Hence the portion GM, of the line LM, lies within GD, and will meet KB if sufficiently produced. Q. E. D.

PROPOSITION VI.

156. *Theorem.*—*Two parallels are everywhere equally distant from each other.*

Dem.—Let E and F be any two points in the line CD, and EG and FH perpendiculars measuring the distances between the parallels CD and AB at these points; then is EG = FH.

For, let P be the middle point between E and F, and PO a perpendicular at this point. Revolve the portion of the figure on the right of PO, upon PO as an axis, until it falls upon the plane of the paper at the left. Then, since FPO and EPO are right angles, PD will fall in PC; and, as PF = PE, F will fall on E. As F and E are right angles, FH will take the direction EC, and H will lie in EC or EC produced. Also, as POH and POC are right angles, OB will fall in OA, and H falling at the same time in EC and OA is at their intersection C. Hence FH coincides with and is equal to EC. Q. E. D.

---

* The learner may think that, if a proposition is true, its converse is necessarily true; and hence, that when a proposition has been proved, its converse may be assumed as also proved. Now this is by no means always the case. Although in a great variety of mathematical propositions, it happens that the proposition and its converse are both true, we never assume one from having proved the other: and we shall occasionally find a proposition whose converse is

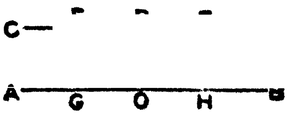


FIG. [illegible].

EXERCISES.

1. ***Prob.****—Through a given point to draw a line parallel to a given line, by the principle contained in* PROP. I. *of this section.*

SUG'S.—Draw a straight line on the blackboard. Designate with a dot some point without the line. To draw a line through the designated point and parallel to the given line, is the problem. Let fall a perpendicular upon the line from the point. Then through the given point draw a line perpendicular to this perpendicular. The latter line will be parallel to the given line. (By what proposition?)

2. ***Prob.****—Through a given point to draw a parallel to a given line by* PROP. III.

SUG'S.—Through the given point draw an oblique line cutting the given line. Then draw a line through the given point making an angle with the oblique line equal to the supplement of the angle which is included between the oblique line and the given line, and on the same side of the former. [Of course the student will be required to *do the work on* the blackboard, *guessing at nothing.*]

Prob.*—Through a given point to draw a line parallel to a given line, upon the principle that the alternate angles made by a secant line are equal* (***152***).

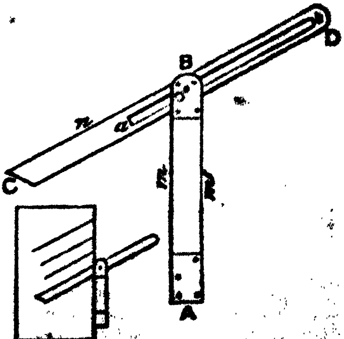


FIG. [illegible].

4. A *bevel* is an instrument much used by carpenters, and consists of a main limb AB, in which a tongue CD is placed, so as to open and shut like the blade of a knife. This tongue turns on the pivot O, which is a screw, and can be tightened so as to hold the tongue firmly at any angle with the limb. The tongue can also be adjusted so as to allow a greater or less portion to extend on a given side, as CB, of the limb. Now, suppose the tongue fixed in position, as represented in the figure, and the side *m* of the limb to be placed against the straight edge of a board, and slid up and down, while lines are drawn along the side *n* of the tongue. What will be the relative position of these lines? Upon what proposition does their relative position depend? How can the carpenter adjust the bevel to a right angle upon the principle in PROP. I., Sec. 1? At what angle is the bevel set, when, drawing two lines from the same point in the edge of the board, one with one edge *m* of the bevel against the edge of the board, and the other with the other edge *m'*, these lines are at right angles to each other?

5. Are the two walls of a building which are carried up by the plumb line exactly parallel? Why?

√ 6. Pass a circumference through three given points, as in (*58*), and show from principles contained in one of the preceding sections, that O is equally distant from A, B, and C; and hence that, if a circumference be drawn from O as a centre with a radius OA, it will pass through A, B, and C.

7. Construct two triangles of unequal sizes, but having the sides of the one respectively parallel to the sides of the other. Are they shaped alike?

8. Construct two triangles of unequal sizes, but having the sides of the one respectively perpendicular to the sides of the other. Are they shaped alike?

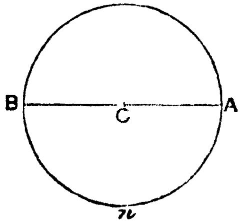
9. Construct a parallelogram, two of whose sides are 6 and 10. Can you construct different-shaped figures with the same sides?

SECTION IV.

OF THE RELATIVE POSITIONS OF STRAIGHT LINES AND CIRCUMFERENCES.

PROPOSITION I.

158. Theorem.*—Any diameter divides a circle, and also its into two equal parts.*



131.

DEM.—Let **AB** be any diameter of the circle **A***m***B***n*; then is the figure **A***m***B** equal to **A***n***B**.

For revolve **A***n***B** upon **AB** as an axis until it falls on the plane of **A***m***B**. Then, since every point in **A***n***B** is at the same distance from the centre **C**, as every point in **A***m***B**, the figures will coincide, and are, consequently, equal. Hence surface **A***n***B** = surface **A***m***B**, and arc **A***n***B** = arc **A***m***B**. Q. E. D.

PROPOSITION II.

159. Theorem.*—A radius which is perpendicular to a chord bisects the chord and also the subtended arc.*

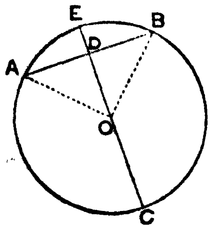


FIG. 132.

DEM.—Let **AB** be any chord and **OE** a radius perpendicular to it at **D**; then **AD** = **BD**, and **AE** = **BE**.*

For, drawing the radii **OA** and **OB**, revolve the semicircle **CBE** upon the diameter **CE** until it falls on **CAE**. The semicircles will coincide (*158*); and since **AB** is perpendicular to **OE**, **DB** will fall in **DA**. Moreover, as there cannot be two equal oblique lines from a point to a line on the same side of a perpendicular, **OB** and **OA** must coincide. Hence **BD** coincides with **AD**, and **BE** with **AE**. Therefore **AD** = **BD**, and **AE** = **BE**. Q. E. D.

---

* To avoid confusing the pupil by a multiplicity of details, the demonstrations in this section are generally limited to the consideration of arcs less than a semi-circumference. All the propositions, except PROP. V., are equally true whatever the arcs, and the demonstrations can easily be applied to cases in which the arcs are greater than semi-circumferences. But this had better not be done till a review is taken, for the reason given above.

160. COR. 1.—*A radius which is perpendicular to a chord bisects the angle subtended by the arc of that chord.*

Thus OE bisects AOB, since BOE is found to coincide with AOE in the demonstration above.

161. COR. 2.—Conversely, *A radius which bisects an arc is perpendicular to the chord of that arc at its middle point.*

DEM.—If OE bisects arc AB at E, when semicircle CBE is revolved on CE till it falls on CAE, EB will coincide with EA; and as D remains fixed and B falls on A, BD coincides with DA. Hence OE has two points, O and D, each equidistant from the extremities of AB, and is, consequently, perpendicular to it at its middle point.

162. COR. 3.—Also, conversely, *A radius which bisects a chord is perpendicular to the chord and bisects the subtended arc.*

For it has two points, each equidistant from the extremities of the chord.

163. COR. 4.—*The line* OD *measures the distance of the chord* AB *from the centre;* since by the distance from a point to a line is always meant the *shortest* distance.

PROPOSITION III.

164. ***Theorem.***—*In the same or in equal circles, equal chords are equally distant from the centre.*

DEM.—Let O and O′ be two equal circles, and chord EF = chord GH; then are the perpendiculars LO and NO′, which measure the distances of the chords from the centre (***163***), equal.

For, since FE is perpendicular to LO and GH to NO′, and LF = NH (***159***), the equal oblique lines FO and HO′ cut off equal distances from the foot of each perpendicular (***141***). Therefore LO = NO′. Q. E. D.

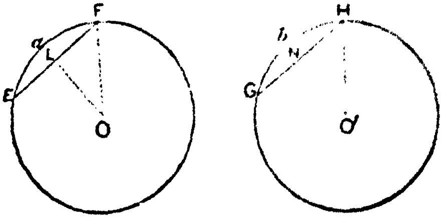


FIG. 1[illegible]

PROPOSITION IV.

165. Theorem.—*In the same or in equal circles, equal arcs have equal chords; and conversely, equal chords subtend equal arcs.*

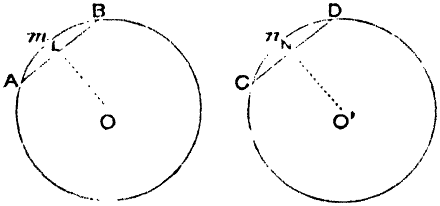


FIG. 134.

DEM.—Let O and O′ be the centres of two equal circles, and arc $AmB =$ arc CnD; then chord $AB =$ chord CD.

Apply the circle O′ to the circle O, with O′ at O, and C at A. Since the circumferences coincide, all the points in each being equally distant from the centre, and since arc $AmB =$ arc CnD by hypothesis, D will fall at B. Hence $AB = CD$.

Conversely, if chord $AB =$ chord CD, arc $AmB =$ arc CnD. Draw the perpendiculars OL and O′N from the centres to the chords. Conceive the plane of circle O′ placed upon circle O, so that CD shall fall upon its equal AB, and O′ be on the same side of AB as O. Since L and N are the middle points of the equal chords, they will coincide; and as LO and NO′ are perpendiculars to the respective chords, and equal (*164*), O′ will fall at O. As the circles are equal, the circumferences will coincide, and consequently the arc AmB coincides with CnD.

PROPOSITION V.

166. Theorem.—*In the same or in equal circles, the less of two arcs has the shorter chord; and, conversely, the shorter chord subtends the less arc.*

DEM.—Let O and O′ be the centres of two equal circles, and arc AmB be less than arc CnD; then is chord $AB <$ chord CD. Drawing the diameters AL and CM, place circle O′ upon circle O, with CM upon AL. Take arc $AD' =$ arc CnD and draw AD′, OB, and OD′. $AD' = CD$ by (*165*). Now $AB < AN + NB$. Also $OD' < ND' + NO$; or, as $OD' = OB$, $OB < ND' + NO$. Subtracting NO from both members, $OB - NO$ (or NB) $< ND'$. Hence, we may substitute ND′ for NB in the inequality $AB < AN + NB$ and have $AB < AN + ND'$ or AD′, which equals CD.

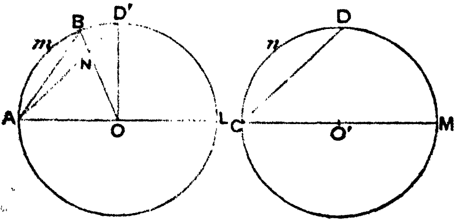


FIG. 135.

Conversely, if chord AB is less than chord CD, arc A*m*B is less than arc C*n*D. For if arc A*m*B = arc C*n*D, chord AB = chord CD (*165*). And, if arc A*m*B > arc C*n*D, chord AB > chord CD. But both of these conclusions are contrary to the hypothesis. Hence, as arc A*m*B can neither be equal to nor greater than arc C*n*D, it must be less.

PROPOSITION VI.

167. Theorem.*—In the same or in equal circles, of two unequal chords, the less is at the greater distance from the centre.*

DEM.—Let CE < AB, then is the perpendicular OD, which measures the distance of CE from the centre, greater than OD' which measures the distance of AB from the centre.

From A lay off AE' = CE, and draw the perpendicular OD''. Then OD'' = OD, since equal chords are equally distant from the centre. As arc AE' < arc AB, AB cuts OD'' in some point as H. Now OH > OD' since the former is oblique and the latter perpendicular to AB. Also OD'' > OH. Much more then is OD'' > OD'. Therefore OD (which equals OD'') > OD'. Q. E. D.

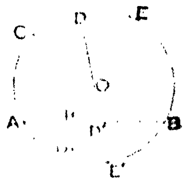


FIG. 136

168. COR.—Conversely, *Of two chords which are unequally distant from the centre, that which is at the greater distance is the less.*

DEM.—Thus, if CE is at a greater distance from the centre than AB, CE < AB. For, if CE were equal to AB, it would be equally distant from the centre. And if CE were greater than AB, it would be at a less distance from the centre. Hence, as CE cannot be at an *equal* distance from the centre with AB, nor at a *less* distance, it must be at a greater.

PROPOSITION VII.

169. Theorem.*—A straight line can intersect a circumference in only two points.*

DEM.—The distances from the centre to the intersections being radii, are equal. Hence, as there can be only two equal straight lines drawn from a point to a straight line, there can be only two intersections. Q. E. D.

PROPOSITION VIII.

170. Theorem.*—A straight line which intersects a circumference in one point intersects it also in a second point.*

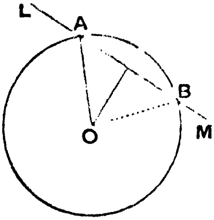


FIG. 137.

DEM.—Let LM intersect the circumference at A; then does it intersect at some other point, as B.

For, since LM intersects the circumference, it passes within it, and has points nearer to O than A. The radius OA is, therefore, an oblique line. Now two equal oblique lines can be drawn from O to the straight line LM. But all points in the plane at the distance OA from O, are in the circumference. Hence there is a second point, as B, common to LM and the circumference. Q. E. D.

171. COR.—*Any line which is oblique to a radius at its extremity, is a secant line.*

PROPOSITION IX.

172. Theorem.*—A straight line which is perpendicular to a radius at its extremity is tangent to the circumference.*

DEM.—The line touches the circumference because the extremity of the radius is in the circumference. Moreover, it does not intersect the circumference, since, if it did, it would have points nearer the centre than the extremity of the radius; but these it cannot have, as the perpendicular is the shortest distance from a point to a line. Hence, as a line which is perpendicular to a radius at its extremity touches the circumference but does not intersect it, it is a tangent (*53*). Q. E. D.

173. COR.—Conversely, *A tangent to a circumference is perpendicular to a radius at the point of contact.*

For, as a tangent to a circumference does not pass within, the point of contact is the nearest point to the centre, and hence is the foot of a perpendicular from the centre.

PROPOSITION X.

174. Theorem.—*Two parallel secants intercept equal arcs.*

Dem.—Let the parallels **LM** and **RS** intersect the circumference **AECF**; then are *the intercepted* arcs **AB** and **DC** equal.

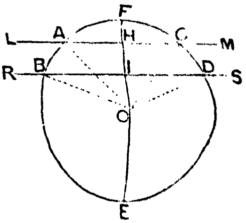


Fig. 138.

Draw the diameter **EF** perpendicular to one of the parallels, as **LM**, whence it will be perpendicular to the other (*144*). Draw the radii **OB** and **OD**. Revolve the portion of the figure on the right of **EF**, upon **EF** until it falls on the plane on the left of **EF**. Then, since **RS** and **LM** are perpendicular to **EF**, **IS** will fall in **IR**, and **HM** in **HL**. Moreover, as there cannot be two equal oblique lines on the same side of a perpendicular, and from the same point (*140*), **OD** and **OB** must coincide, and **D** fall at **B**. In like manner **C** falls at **A**, and **CD** coincides with **AB**. Therefore **CD** = **AB**. Q. E. D.

PROPOSITION XI.

175. Theorem.—*If a secant be parallel to a tangent, the arcs intercepted between the intersections and the point of tangency are equal.*

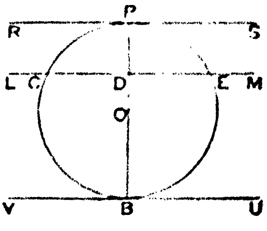


Fig. 139.

Dem.—Let the secant **LM** be parallel to the tangent **RS**; then is **CP** = **EP**.

For, draw the radius **OP** to the point of tangency; it will be perpendicular to the tangent (*173*), and also to the parallel **LM** (*144*). But a radius which is perpendicular to a chord, as **OP** to **CE**, bisects the subtended arc (*159*), hence **CP** = **EP**. In like manner, if **VU** is parallel to **LM**, **CB** = **EB**. Q. E. D.

176. Cor.—*Two parallel tangents include equal arcs between the points of tangency; and these arcs are semi-circumferences.*

EXERCISES.

1. Draw a circle and divide it into two equal parts. What proposition is involved?

2. Given a point in a circumference, to find where a semi-circumference reckoned from this point terminates. What proposition is involved?

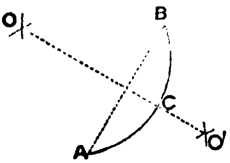


Fig. 140.

3. ***Prob.***—*To bisect a given arc.*

Solution.—Let AB be an arc which we wish to bisect.[*] Draw its chord AB, and erect OO′ bisecting the chord, by (***130***). Now, as OO′ is perpendicular to the chord at its middle point, it bisects the arc by (***162***), since there can be but one perpendicular at the middle point of the chord. The arc AB is, therefore, bisected at C, *i.e.*, AC = CB.

4. ***Prob.***—*To bisect a given angle.*

Sug.—The method of solving this is given in Part I. The student should *do it* as there directed, and then point out the principle upon which the method depends.

5. In a circle whose radius is 11 there are drawn two chords, one at 6 from the centre, and one at 4. Which chord is the greater? By what proposition?

6. In a certain circle there are two chords, each 15 inches in length. What are their relative distances from the centre? Quote the principle.

7. There is a circular plat of ground whose diameter is 20 rods. A straight path in passing runs within 7 rods of the centre. What is the position of the path with reference to the plat? What is the position of a straight path whose nearest point is 10 rods from the centre? One whose nearest point is 11 rods from the centre?

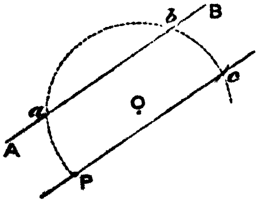


Fig. 141.

8. Pass a line through a given point, and parallel to a given line, by the principles contained in (***174***) and (***165***).

---

[*] This solution and many others are given, not so much that it is feared that the student will not be able to solve the problems, as to afford models for describing the process. In this case an arc should be drawn first, and all trace of the centre obliterated. *Then* proceed as directed.

9. ***Prob.****—To draw a tangent to a circle at a given point in the circumference.*

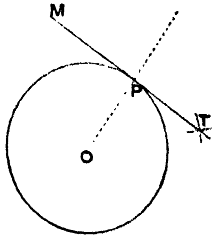


FIG. 141*.

SOLUTION.—Let P be the point at which a tangent is to be drawn. Draw the radius OP to the given point of tangency, and produce it any convenient distance beyond the circle. Erect a perpendicular to this line at P, as MT; then is MT a tangent to the circle (*172*).

10. ***Prob.****—To find the centre of a circle whose circumference is known, or of any arc of it.*

SUG.—The *process* is given in PART I. Do the work as there directed, and then show upon what proposition in this section it is founded.

SYNOPSIS.

RELATIVE POSITIONS OF STRAIGHT LINES AND CIRCUMFERENCES.	DIAMETERS.	PROP. I.	How divide circles and circumferences.	
	CHORDS.	PROP. II.	Radius perp. to chord.	*Cor.* 1. Bisects angle *Cor.* 2. Converse of Prop. *Cor.* 3. " " " *Cor.* 4. Dist. from centre.
		PROP. III.	Distance of equal chords from centre.	
		PROP. IV.	Equal arcs, and converse.	
		PROP. V.	Unequal arcs.	
		PROP. VI.	Unequal chords. Distance from centre.	*Cor.* Converse.
	SECANTS.	PROP. VII.	Intersect in only two points.	
		PROP. VIII.	If a line intersect in one point, it intersects also in another.	*Cor.* Line oblique to radius at extr.
	TANGENTS.	PROP. IX.	Line perpendicular to radius at extremity.	*Cor.* Converse.
	PARALLELS.	PROP. X.	Parallel secants intercept equal arcs.	
		PROP. XI.	Secant par. to tangent.	*Cor.* Two parallel tangents.
	EXERCISES.	*Prob.*	To bisect an arc.	
		Prob.	To bisect an angle.	
		Prob.	To draw a tangent at a point in circumference.	
		Prob.	To find centre of circumference or arc.	

SECTION V.

OF THE RELATIVE POSITIONS OF CIRCUMFERENCES.

PROPOSITION I.

177. Theorem.—*All the circumferences which may be passed through three points not in the same straight line coincide, and are one and the same.*

Dem.—Let A, B, and C be three points not in the same straight line; then all the circumferences which can be passed through them will coincide.

For join the points, two and two, by straight lines, as AB and BC. Bisect these lines with perpendiculars, as DF and EH. Since AB and BC are not in the same straight line, DF and EH will meet when sufficiently produced, at one and only one point, as O, because they are straight lines Now, every point in FD is equally distant from A and B, and every point in HE is equally distant from B and C (*129*). Hence O is equally distant from the three points A, B, and C; and, if a circumference be drawn with O as a centre, and a radius AO, it will pass through the three points. Moreover, every circumference passing through these points must have O for its centre, since the centre must be in FD (otherwise it would be unequally distant from A and B), and also in HE (*129*). But these lines intersect only in O. Also, every circumference with O as its centre, and passing through A, must have AO for its radius. Hence, as all circles having the same centre and the same radius coincide, all those passing through three points, A, B, and C, coincide. Q. E. D.

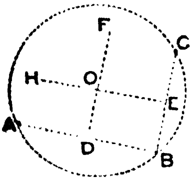


Fig. 142.

178. Cor. 1.—*Through any three points not in the same straight line a circumference can be passed.*

179. Cor. 2.—*Three points not in the same straight line determine a circumference as to position and extent;* i.e., *in all respects.*

180. Cor. 3.—*Two circumferences can intersect in only two points.*

For, if they have three points common, they coincide, and form one and the same circumference.

PROPOSITION II.

181. *Theorem.*—*Two circumferences which intersect in one point, intersect also in a second point.*

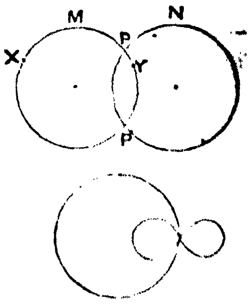


Fig. 143.

Dem.—Let M intersect N at P. As M passes from without to within the circle N, it has points both without and within. Now, for M to return into itself from any point within N, as Y, to any point without, as X, it must intersect N; but it cannot intersect in P, for a circumference does not intersect itself. Hence, it intersects in a second point, as P'. Q. E. D.

PROPOSITION III.

182. *Theorem.*—*If a straight line be drawn through the centres of two circles, of the intersections of either circumference with that line, the one on the side toward the centre of the other circle is the nearest point in this circumference to that centre, and the one on the opposite side is the farthest point from that centre.*

Dem.—Let M and N, or M' and N', be two circumferences whose centres are O and O'. Draw an indefinite line through these centres. Let A and H be the intersections of M or M' with this line, of which A is on the side of M or M' toward the centre O', and H is on the opposite side. Then is A the nearest point in M or M' to O', and H the farthest point from O'.

First, To show that A is nearer O' than any other point in the circumference. A will lie between O and O', in O', or beyond O'. When A lies between O and O', as in M, let P be any other point in M, and draw OP and O'P. Now OO' being a straight line, is less than OPO', a broken line. Subtracting OA from the former, and its equal OP from the latter, we have $AO' < PO'$. When A falls at O' the truth is self-evident. When A lies beyond O', as in M', let P be any other point in M', and draw OP and O'P. Now $O'P + OO' > OP$ $(= OA)$. Subtracting OO' from both, we have $O'P > OA - OO'$ $(= O'A)$. Hence, in any case, A is the nearest point in M or M' to O'.

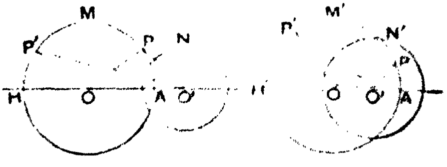


Fig. 144.

Second, To show that H is the farthest point in M or M' from O'. In either figure, let P' be any other point in the circumference than H, and draw OP' and $O'P'$. Now, $P'O + OO' > P'O'$. But $P'O = HO$. Hence $HO + OO'$ ($= HO'$) $> P'O'$

PROPOSITION IV.

183. *Theorem.*—*When the distance between the centres of two circles is greater than the sum of their radii, the circumferences are wholly exterior the one to the other.*

DEM.—Let M and N be the circumferences of two circles whose centres are O and O'. Let OO' be greater than the sum of the radii. Then are M and N wholly exterior the one to the other.

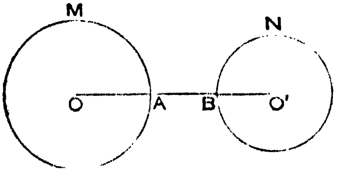


FIG. 145.

For A, the intersection of M with OO', is between O and O', since $OA < OO'$. Now, by hypothesis, $OO' > OA + BO'$. Subtracting OA from both, we have $AO' > BO'$. Hence, as the nearest point in M is farther from O' than the circumference of the latter circle, M lies wholly exterior to N. Q. E. D.

184. COR.—Conversely, *When two circumferences are exterior the one to the other, the distance between their centres is greater than the sum of their radii.*

DEM.—For, join the centres OO' with a straight line. Now the point A where this line cuts the circumference M is the nearest point in this circumference to the centre O'. But, by hypothesis, this (and every other point in circumference O) is without circle O'. Hence, $AO' > BO'$. To each add OA, and $OA + AO'$ (or OO') $> OA + BO'$.

PROPOSITION V.

185. *Theorem.*—*When the distance between the centres of two circles is equal to the sum of their radii, the circumferences are tangent to each other externally.*

DEM.—Let M and N be two circumferences, and OO', the distance between their centres, be equal to $OC + O'C'$, the sum of their radii; then are the circumferences tangent to each other externally.

The point A, where M cuts the line joining the centres, is between O and O′, since $OA < OO'$ by hypothesis. Moreover, A is the nearest point in M to the centre O′. Again, as $OO' = OC + O'C'$, subtracting OA from the first member, and its equal OC from the other, we have $O'A = O'C'$; that is, A is in the circumference N. Hence, as A lies in N, and all other points in M are more distant from O′ than the length of the radius O′C′, M is entirely without N, except the point A, and the circles are *tangent to each other externally*. Q. E. D.

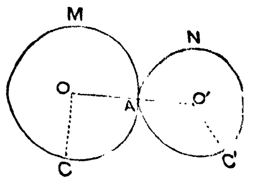


FIG. 146.

186. COR. 1.—Conversely, *When two circumferences are tangent to each other externally, the distance between their centres is equal to the sum of their radii.*

DEM.—M being tangent to N externally, the point in M nearest the centre O′ must be in N, while all other points in M are exterior to N. Now, the point in M nearest to O′ is A on the line joining their centres (*182*). A is therefore the point of tangency, and $OO' = OA + O'A$.

187. COR. 2.—*When two circumferences are tangent to each other externally, the point of tangency is in the line joining their centres.*

PROPOSITION VI.

188. ***Theorem.***—*When the distance between the centres of two circles is less than the sum and greater than the difference of their radii, the two circumferences intersect.*

DEM.—Let M and N be the circumferences of two circles whose centres are O and O′. Let the radius of M be equal to or greater than the radius of N. Now, if $OO' < OA + O'B$, and $> OA - O'B$, M and N intersect.

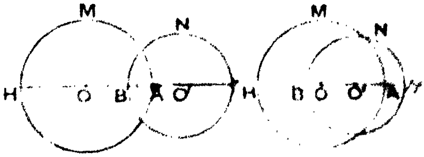


FIG. 147.

For, when $OO' > OA$, $OO' < OA + O'B$ gives $OO' - OA$ $(= AO') < O'B$; and when $OO' \overline{<} OA$, $OO' > OA - O'B$ gives $O'B > OA - OO'$ $(= AO')$. Hence the nearest point in M to O′ lies within N. Again, to the first member of $OO' > OA - O'B$ add HO, and to the second its equal OA, and we have $OO' + HO$ $(= HO') > 2OA - O'B$. Now, since $O'B \overline{<} OA$,* by hypothesis, the difference $2OA - O'B \overline{>} O'B$. Hence, $HO' > O'B$, and H lies without N. As, therefore, M has one point at least within N and one without, M and N intersect. Q. E. D.

---

* Read "O′B is equal to or less than OA."

189. COR.—Conversely, *When two circumferences intersect, the distance between their centres is less than the sum and greater than the difference of their radii.*

DEM.—Let the radius of N be equal to or less than the radius of M. As the circumferences intersect the farthest point H′ of N from O must be farther from O than the length of the radius of M, *i. e.*, must lie without that circle. So we have by hypothesis H′O > OA. Subtracting H′O′ from the first member and its equal BO′ from the second, we have H′O − O′H′ (= OO′) > OA − BO′; that is, the distance between the centres is greater than the difference of the radii. Again, as the nearest point in M to O′ must lie within N, we have AO′ < BO′, and adding OA to both members, OA + AO′ (= OO′) < OA + BO′; that is, the distance between the centres is less than the sum of the radii.

PROPOSITION VII.

190. Theorem.—*When the distance between the centres of two unequal circles is equal to the difference of their radii, the less circle is tangent to the other internally.*

DEM.—Let M and N be the circumferences of two circles whose centres O and O′ are so situated that OO′ = OC − O′C′; then are the circles tangent to each other internally.

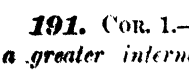


FIG. 148.

For, let N be the circumference of the less circle, so that OC > O′C′. Let HH′ be a diameter of M. By hypothesis OO′ = OC − O′C′. Now, subtracting each member of this equality from OH′, we have OH′ − OO′ (= O′H′) = O′C′. Whence it appears that H′, the point in N at the greatest distance from O, is in M; and, consequently, that every other point in N is within M. Hence, N is tangent to M internally. Q. E. D.

191. COR. 1.—Conversely, *When a less circumference is tangent to a greater internally, the distance between their centres equals the difference of their radii.*

less circumference N being tangent to the greater M, internally, the point in N at the greatest distance from the centre O of [illegible], must be in M, while all other points of N lie within M. Now H′ in the line passing through the centres is the point of N at the greatest distance from O. Hence we observe that OO′ = OH′ − O′H′, *i. e.*, the difference between the radii.

192. Cor. 2.—*When one circumference is tangent to another internally, the point of tangency is in the line passing through their centres.*

193. Sch.—If the radii are equal the two circumferences coincide.

PROPOSITION V

194. Theorem.—*When the distance between the centres of two unequal circles is less than the difference of their radii, the less circumference is wholly within the greater.*

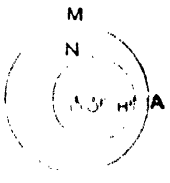


Fig. 119

Dem.—Let N be a less circumference than M, and OO', the distance between their centres, be less than $OA - O'H'$, the difference of their radii; then is N wholly within M.

For, to each member of $OO' < OA - O'H'$ add $O'H'$, and we have $OO' + O'H' < OA$. But $OO' + O'H' = OH'$. Hence $OH' < OA$, and H', the farthest point in N from O, is within M, and consequently N lies wholly within M. Q. E. D.

195. Cor.—Conversely, *When a less circumference is wholly within a greater, the distance between their centres is less than the difference of their radii.*

Dem.—If N lies wholly within M, the farthest point in N from O, the centre of M, must be nearer O than is any point in M, *i. e.*, $OH' < OA$. Now, subtract $O'H'$ from each member, and we have $OH' - O'H' \ (= OO') < OA - O'H'$. Q. E. D.

196. Sch.—If the centres coincide so that $OO' = 0$, the circumferences are said to be *concentric*. If, at the same time, their radii are equal, they are *coin-*

IX.

197. Theorem.—*When two circumferences intersect, the line which passes through their centres is perpendicular to their common chord at its middle point.*

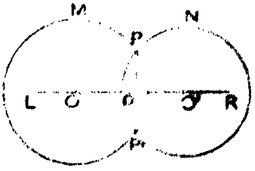


Fig. 120.

Dem.—Let the circumferences M and N intersect in the points P and P' (*181*); let PP' be the common chord, and LR the line passing through the centres O and O'; then is LR perpendicular to PP'.

For O′ is equally distant from the extremities P and P′, and O is also equally distant from P and P′. Hence, as LR has two points equally distant from the extremities of PP′, it is perpendicular to PP′ at its middle point. Q. E. D.

PROPOSITION X.

198. Theorem.—*When one circumference is tangent to another, either externally or internally, they have a common rectilinear* tangent at their common point.*

DEM.—Since the radii of the two circles drawn to the common point form one and the same straight line (*187, 192*), a line perpendicular to one at its extremity is perpendicular to the other also. And a line which is perpendicular to a radius at its extremity is tangent to the circle (*172*). Q. E. D.

199. COR.—*All circumferences having their centres in the same line, and having but one common point, are tangent to each other, and have a common rectilinear tangent at the common point.*

EXERCISES.

1. **Prob.**—*To pass a circumference through three given points not in the same straight line.*

SUG.—The process should be gone through with as learned from PART I., and then the reasons for the process given as furnished by this section.

2. To pass a circumference through two given points, whose center shall be in a given line.

3. **Prob.**—*To circumscribe a circumference about a given triangle, and give the reasons for the process.*

4. The centres of two circles whose radii are 10 and 7, are at 4 from each other. What is the relative position of the circumferences? What if the distance between the centres is 17? What if 20? What if 2? What if 0? What if 3?

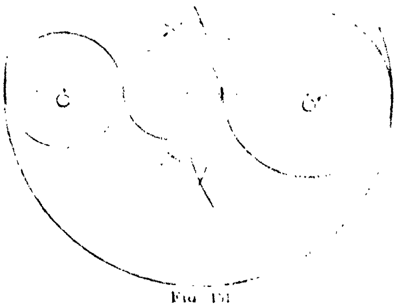


FIG. 94

5. Given two circles O and O', to draw two others, one of which shall be tangent to these externally, and to the other of which the two given circles shall be tangent internally. Give *all* the principles involved in the construction. Give other methods.

6. Given two circles whose radii are 6 and 10, and the distance between their centres 20. To draw a third circle whose radius shall be 8, and which shall be tangent to the two given circles? Can a third circle whose radius is 2 be drawn tangent to the two given circles? How will it be situated? Can one be drawn tangent to the given circles, whose radius shall be 1? Why?

SYNOPSIS.

RELATIVE POSITIONS OF CIRCUMFERENCES.

- PROP. I. Through three points.
  - *Cor.* 1. A circf. can be passed.
  - *Cor.* 2. A circf. determined by
  - *Cor.* 3. Intersections of two circf's
- PROP. II. Two circumferences which intersect in *one* point.
- PROP. III. Points in one circumference nearest to and farthest from the centre of another.
- DISTANCE BETWEEN CENTRES OF TWO CIRCLES.
  - PROP. IV. Greater than sum of radii. { *Cor.* Converse.
  - PROP. V. Equal to sum of radii.
    - *Cor.* 1. Converse.
    - *Cor.* 2 Point of tangency.
  - PROP. VI. Less than sum and greater than difference of radii. { *Cor* Converse.
  - PROP. VII. Equal to diff. of radii.
    - *Cor.* 1. Converse.
    - *Cor.* 2. Point of tangency.
    - *Sch.* Radii equal.
  - PROP. VIII. Less than diff. of radii.
    - *Cor.* Converse.
    - *Sch.* Concentric, Coincident.
- PROP. IX. Perpendicular to common chord.
- PROP. X. Common tangent to two circles tangent to each other. } *Cor.* To all.
- EXERCISES.
  - *Prob.* To pass circumference through three points.
  - *Prob.* To circumscribe a triangle with a circumference.

SECTION VI.

OF THE MEASUREMENT OF ANGLES.

200. Angles are said to be measured by arcs, according to the principles developed in the three following propositions.

PROPOSITION I.

201. ***Theorem.****—In the same or in equal circles, equal arcs subtend equal angles at the centre.*

DEM.—In the equal circles **M** and **N**, let *arc* **AB** = *arc* **DC**; then will the angles **O** and **O′**, called angles at the centre, be equal. For, placing **N** upon **M** so that **O′** shall fall on **O**, and **O′D** on **OA**, since the circles are equal, **D** will fall on **A**; and since, by hypothesis, *arc* **DC** = *arc* **AB**, **C** will fall on **B**. Hence, **O′C** will coincide with **OB**, and *angle* **O′** = *angle* **O**, because they coincide when applied. Q. E. D.

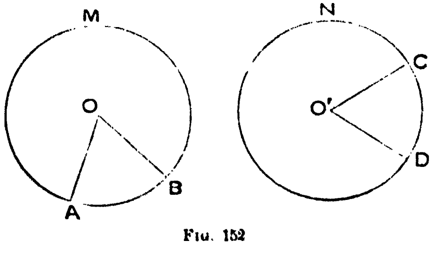


FIG. 152

202. COR. 1.—Conversely, *In the same or in equal circles, equal angles at the centre intercept equal arcs.*

DEM.—If, by hypothesis, *angle* **O′** = *angle* **O**, in the equal circles **M** and **N**, *arc* **DC** = *arc* **AB**. For, placing circle **N** upon **M**, so that **O′** shall fall on **O**, and **O′D** on its equal **OA**, **D** falls on **A**, and, since *angle* **O′** = *angle* **O**. **O′C** takes the direction **OB**, and, being equal to it, **C** falls on **B**. Hence, **DC** and **AB** coincide and are equal.

203. COR. 2.—*A right angle at the centre intercepts a quarter of a circumference, and is said to be measured by it. Hence, a semicircumference is the measure of two right angles, and a whole circumference of four.*

PROPOSITION II.

204. Theorem.—*In the same or in equal circles, arcs which are in the ratio of two whole numbers subtend angles at the centre which have the same ratio, whence the angles are to each other as the arcs which subtend them.*

DEM.—In the equal circles M and N, let the arcs EF and IH, which subtend the angles O and O′ at the centre, be in the ratio of 5 to 8; then are the angles O and O′ in the ratio of 5 to 8, and we have

angle O : *angle* O′ :: *arc* EF : *arc* IH.

For, divide EF into 5 equal parts, as E*a*, *ab*, etc., then IH can be divided into 8 such parts, I*e*, *ef*, etc. Draw the radii O*a*, O*b*, O*c*, etc., and O′*e*, O′*f*, O′*g*, etc.; and, since these partial arcs are equal, the partial angles which they subtend are equal, by the preceding proposition. Now, O is composed of 5 of these angles, and O′ of 8; whence

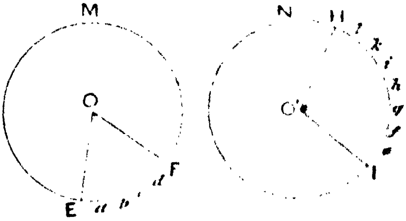


FIG. 153

angle O : *angle* O′ :: 5 : 8.

But, *arc* EF : *arc* IH :: 5 : 8.

Hence, the two ratios being equal, we have

angle O′ : *angle* O :: *arc* IH : *arc* EF.

As the same method could be pursued in case the arcs were to each other as any other two whole numbers, the argument is general.

205. COR.—Conversely, *In the same or in equal circles, angles at the centre which are in the ratio of two whole numbers are to each other as their intercepted arcs.*

DEM.—Thus, let *angle* O′ be to *angle* O in the ratio of 8 to 5. Conceive O′ divided into 8 equal partial angles, then will O be divisible into 5 such partial angles. Now, the partial angles being equal, their intercepted arcs are equal, by the preceding proposition, *Cor* 1. Whence,

arc IH : *arc* EF :: 8 : 5.

But, *angle* O′ : *angle* O :: 8 : 5, by hypothesis.

Hence, *arc* IH : *arc* EF :: *angle* O′ : *angle* O.

And the same method could be pursued with angles having the ratio of any other whole numbers.

PROPOSITION III.

206. *Theorem.*—*In the same circle or in equal circles, two incommensurable arcs are to each other as the angles which they subtend at the centre.*

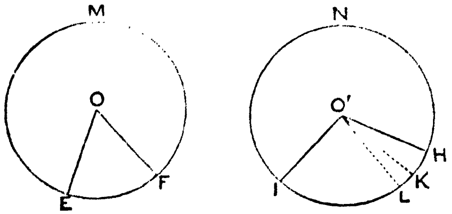


FIG. 154.

DEM.—In the equal circles M and N, let EF and IH be incommensurable arcs. Now there is some arc to which EF bears the same ratio as *angle* O to *angle* O′. If that arc is not IH let it be IL, an arc *less* than IH, so that

angle O : *angle* O′ : : *arc* EF : *arc* IL.[*]

Conceive EF divided into equal parts, each of which is less than LH,[†] the assumed difference between IH and IL. Then conceive one of these equal parts to be applied to IH as a measure, beginning at I. Since the measure is less than LH, at least one point of division must fall between L and H. Suppose K to be such a point. Draw O′K. Now, the arcs EF and IK are commensurable, and by the last proposition

angle O : *angle* IO′K : : *arc* EF : *arc* IK. But we assumed that

angle O : *angle* IO′H : : *arc* EF : *arc* IL.

In these proportions the antecedents being alike, the consequents should be proportional, so that

angle IO′K should be to *angle* IO′H : : *arc* IK : *arc* IL.

But this proportion is false, since

angle IO′K $<$ *angle* IO′H, whereas *arc* IK $>$ *arc* IL.

In a manner altogether similar (the student should supply it) we can show that

angle O is not to *angle* O′ : : *arc* EF : any *arc greater* than IH.

Hence, as the fourth term of the proportion cannot be less or greater than IH, it must be IH itself; and

angle O : *angle* O′ : : *arc* EF : *arc* IH. Q. E. D.

207. COR.—Conversely, *In the same or in equal circles, two incommensurable angles at the centre are to each other as the arcs which they intercept.*

---

[*] This is a false hypothesis, and the object of the argument following is to show its falsity.

[†] This can be done by supposing EF bisected, then the halves bisected, then the fourths bisected, and this process of bisection continued until the parts are each less than LH.

magnitude of an angle from the statement of the number of degrees which measure it; and, for brevity, the angle is spoken of as an angle of the same number of degrees as the intercepted arc.

---

209. An Inscribed Angle is an angle whose vertex is in a circumference, and whose sides are chords, or a chord and diameter, of that circumference.

PROPOSITION IV.

210. Theorem.*—An inscribed angle is measured by half the arc intercepted between its sides.*

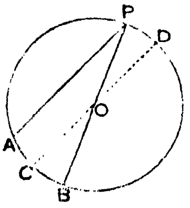


Fig. 155.

Dem.—*First, when one side is a diameter.* Let **APB** be an inscribed angle, and **PB** a diameter; then is **APB** measured by one-half of arc **AB**. For, through the centre **O**, draw the diameter **DC** parallel to the chord **PA**; then **COB** = **POD** (***134***), whence *arc* **CB** = *arc* **PD** (***202***), also **COB** = **APB** (***152***); and *arc* **PD** = *arc* **AC** (***174***), whence **PD** = **CB** = ½**AB**. Now **COB** is measured by **CB** (***208***); hence **APB** is measured by **CB** = ½**AB**. Q. E. D.

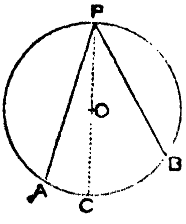


Fig. 156.

Second, when both sides are chords and the centre of the circle lies between them. Let **APB** be such an angle. Draw the diameter **PC**. Now, by the preceding part, **APC** is measured by ½**AC**, and **CPB** by ½**CB**. Hence **APC** + **CPB**, or **APB**, is measured by ½**AC** + ½**CB**, or ½**AB**. Q. E. D.

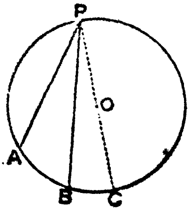


Fig. 157.

Third, when both sides are chords and the centre lies without the angle. Let **APB** be such an angle. Draw the diameter **PC**. Now **APC** is measured by ½**AC**, and **BPC** by ½**BC**. Hence **APC** − **BPC**, or **APB**, is measured by ½**AC** − ½**BC**, or ½**AB**. Q. E. D.

211. COR.—*In the same or equal circles all angles inscribed in the same segment, intercept equal arcs, and are consequently equal. If the segment is less than a semicircle, the angles are obtuse; if a semicircle, right; if greater than a semicircle, acute.*

ILL.—In each separate figure the angles **P** are equal, for they are each measured by half the same arc. . . . In **O**, each angle **P** is acute, being measured by $\frac{1}{2}m$, which is less than a quarter of a circumference. . . . In **O**$'$, each angle **P** is a right angle, being measured by $\frac{1}{2}m'$, which is a quadrant (quarter of a circumference). . . . In **O**$''$, each angle **P** is obtuse, being measured by $\frac{1}{2}m''$, which is greater than a quadrant.

SCH.—The converse of this proposition is usually taken for granted; *i.e.*, that if the several angles **P**, **P**, etc., are equal and subtended by the same chord, their vertices lie in the circumference. This is readily proved rigorously after the next two propositions. Thus, if vertex **P** were without, the angle would be measured by ½**AB** − ½ the other intercepted arc; and if within, by ½**AB** + ½ the other intercepted arc.

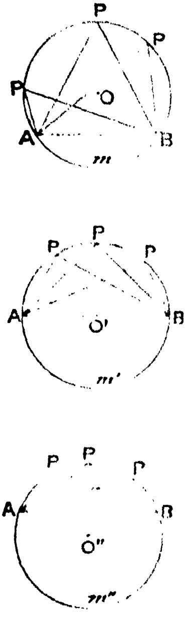


FIG. 158.

PROPOSITION V.

212. Theorem.—*Any angle formed by two chords intersecting in a circle is measured by one-half the sum of the arcs intercepted between its sides and the sides of its vertical, or opposite, angle.*

DEM.—Let the chords **AB** and **CD** intersect at **P**; then is **APD**, or its equal **CPB**, measured by ½(**AD** + **CB**); and **APC**, or its equal **BPD**, is measured by ½(**AC** + **BD**).

For, through **C** draw **CE** parallel to **BA**; whence **ECD** = **APD** (***152***), and **CB** = **EA** (***174***). But **ECD** is measured by ½ **ED** (***210***), which equals ½ (**AD** + **EA**) = ½(**AD** + **CB**).

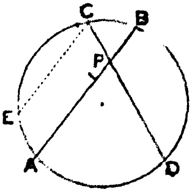


FIG. 159.

That **APC**, or its equal **BPD**, is measured by ½(**AC** + **BD**), appears from the fact that the sum of the four angles about **P** being equal to four right angles, is measured by a whole circumference (***208***). But **APD** + **CPB** is measured by **AD** + **CB**; whence **APC** + **BPD**, or 2 **APC**, is measured by the whole circumference minus (**AD** + **CB**); that is, by **AC** + **BD**. Then is **APC** measured by ½(**AC** + **BD**).

213. SCH.—The case of the angle included between two chords passes into that of the inscribed angle in the preceding proposition, by conceiving AB to move parallel to its present position until P arrives at C and BA coincides with CE. The angle APD is all the time measured by half the sum of the intercepted arcs; but, when P has reached C, CB becomes 0, and APD becomes an inscribed angle measured by half its intercepted arc.

In a similar manner we may pass to the case of an angle at the centre, by supposing P to move toward the centre. All the time APD is measured by $\frac{1}{2}$(AD + CB); but, when P reaches the centre, AD = CB, and $\frac{1}{2}$(AD + CB) = $\frac{1}{2}$(2AD) = AD; *i. e.*, an angle at the centre is measured by its intercepted arc.

PROPOSITION VI.

214. Theorem.*—An angle formed by two secants meeting without the circle is measured by one-half the difference of the intercepted arcs.*

DEM.—Let APB be an angle formed by the two secants AP and PB; then is it measured by $\frac{1}{2}$(AB − CD), *i. e.*, one-half the intercepted arcs.

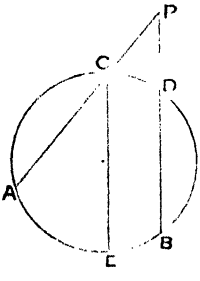


FIG. 160.

For, draw CE parallel to PB; then CD = EB (*174*), and ACE = its corresponding angle APB. But ACE is measured by $\frac{1}{2}$ AE − $\frac{1}{2}$ (AB − EB) = $\frac{1}{2}$ (AB − CD). Q. E. D.

215. SCH.—This case passes into that of an inscribed angle, by conceiving P to move toward C, thus diminishing the arc CD. When P reaches C, the angle becomes inscribed; and, as CD is then 0, $\frac{1}{2}$(AB − CD) = $\frac{1}{2}$ AB. Also, by conceiving P to continue to move along PA, CD will reappear *on the other side of* PA, hence will change its sign,* and $\frac{1}{2}$(AE − CD) will become $\frac{1}{2}$(AE + CD), as it should, since the angle is then formed by two chords intersecting within the circumference.

PROPOSITION VII.

216. Theorem.*—An angle formed by a tangent and a chord drawn from the point of tangency is measured by one-half the intercepted arc.*

---

* In accordance with the law of positive and negative quantities as used in mathematics, whenever a continuously varying quantity is conceived as diminishing till it reaches 0, and then as reappearing by the same law of change, it must change its sign.

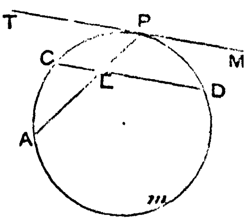


FIG. 161.

DEM.—Let TPA be an angle formed by TM tangent at P, and the chord PA; then is TPA measured by one-half the intercepted arc AP. For, draw any chord CD parallel to TM and cutting AP. Then CEA = TPA. But CEA is measured by $\frac{1}{2}$ (AC + PD). Hence, as PD = CP (*175*), TPA is measured by $\frac{1}{2}$ (AC + CP), or $\frac{1}{2}$ AP. Q. E. D.

Show that APM is measured by $\frac{1}{2}$ arc A*m*P.

Also, observe how the case of two secants (*214*) passes into this.

PROPOSITION VIII.

217. Theorem.—*An angle formed by two tangents is measured by one-half the difference of the intercepted arcs.*

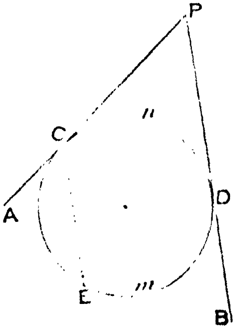


FIG. 162.

DEM.—Let APB be an angle formed by the two tangents AP and PB; then is it measured by $\frac{1}{2}$ (*arc* C*m*D − *arc* C*n*D), *i. e.*, one-half the difference of the intercepted arcs. For, through one of the points of tangency, as C, draw a chord, as CE, parallel to the other tangent. Now, ACE is measured by $\frac{1}{2}$ *arc* CE, by the last proposition. But ACE = APB, and *arc* CE = C*m*D − D*m*E = C*m*D − C*n*D, since C*n*D = E*m*D (*175*). Hence, APB is measured by $\frac{1}{2}$ (C*m*D − C*n*D). Q. E. D.

218. SCH.—The case of two secants (*214*) passes into this by supposing the secants to separate until they become tangents.

EXERCISES.

1. **Prob.**—*Through a given point to draw a parallel to a given line, on the principles contained in (149), (201), and (165).*

SOLUTION.—Through P to draw a parallel to AB. From P as a centre, with any radius greater than the distance from P to AB, describe an arc cutting AB, as *ac*. From *a* as a centre, with the same radius, strike an arc through P, intersecting AB, as P*b*. Take the chord P*b* and apply it from *a* on the arc *ac*, as *a*O. These chords being equal, the arcs P*b* and

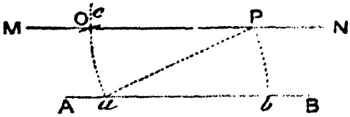


FIG. 163.

aO are equal (*165*). Again, angle Pab = angle OPa, since they are measured by equal arcs struck with the same radius (*201*). These alternate angles being equal, MN is parallel to AB (*149*).

2. In *Fig.* 164 there are 4 pairs of equal angles. Which are they, and why? Show also that COB = ABD + CDB, by (*210*), and (*212*). Show also that DOB = ABC + DAB.

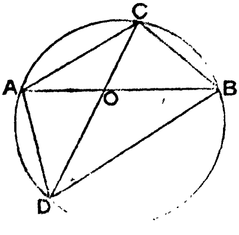


Fig. 164.

3. ***Prob.***—*From a point without a circle to draw a tangent to the circle.*

Solution.—Let O be the given circle, and P the given point. Join P with the centre O, and upon PO as a diameter describe a circle. Let T and T′ be the intersections of the two circumferences. Now, if lines be drawn from P through T and T′, they will be tangent to the circle O. For OTP and OT′P, being inscribed in semi-circles, are right angles (*211*). Hence, PM is perpendicular to radius OT at its extremity T, and is therefore a tangent (*172*). In like manner PT′ is shown to be a tangent, and we see that *from a point without a circle two tangents can be drawn to the circle.*

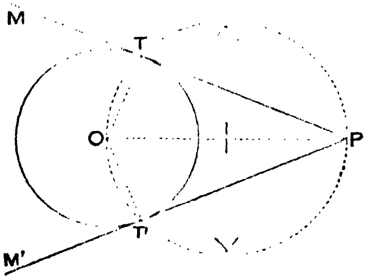


Fig. 165.

4. ***Prob.***—*On a given line, to construct a segment which shall contain a given angle.*

Solution.—Let AB be the given line, and O the given angle. At one extremity of the given line, as B, construct an angle ABC equal to the given angle O, which shall lie on the opposite of AB from that on which the required segment is to lie. Erect a perpendicular to the line CB at B, and also a perpendicular bisecting AB. Let FB and FE be

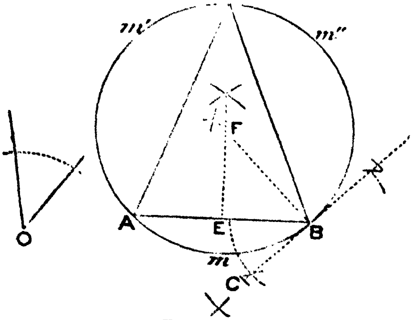


Fig. 166.

these perpendiculars, intersecting at F. From F as a centre, with a radius equal to FB, describe a circle. Then is AHB the segment required. For, CB being perpendicular to radius FB at its extremity, is tangent to the circle, and *angle* ABC (= *angle* O) is measured by ½ of arc A*m*B (**216**). Now, any angle inscribed in the segment A*m'm'*B, as AHB, has ½ A*m*B for its measure, and is, consequently, equal to O.

Another Solution.—On the side of AB on which the segment is to lie, draw *any* line through either extremity of AB, *making an acute angle* with AB. Let CB be such a line. At *any* point in CB, as C, draw a line CE, making angle ECB = the given angle O, *Fig.* 166. Through A pass a parallel to CE (see *Ex.* 1), as AD. Pass a circumference through A, D, and B. Any angle inscribed in segment A*m*B is equal to O. [Let the student give all the reasons, and make the construction. The requisite marks for the construction are made in the figure. Why is it said, make CBA an *acute angle?* When would a right angle answer? When an obtuse angle?]

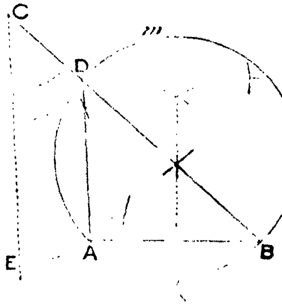


Fig. 167.

SYNOPSIS.

Measurement of Angles.

- How angles are measured.
- Fundamental Propositions.
  - Prop. I. Equal arcs subtend equal angles at the centre.
    - *Cor.* 1. Converse.
    - *Cor.* 2. Measure of 1, 2, and 4 right angles.
  - Prop. II. Commensurable arcs in the same ratio as their subtended angles.
    - *Cor.* Converse.
  - Prop. III. Incommensurable arcs.
    - *Cor.* Converse.
    - *Sch.* Method of measuring angles.
- Inscribed angle, what?
- Prop. IV. Inscribed angle, how measured.
  - *Cor.* In segment, $>$, $=$, or $<$ semicircle.
- Prop. V. Angle between two chords.
  - *Sch.* Compared with preceding.
- Prop. VI. Angle between two secants.
  - *Sch.* Compared with Prop. IV.
- Prop. VII. Angle between tangent and chord.
- Prop. VIII. Angle between two tangents.
  - *Sch.* Compared with Prop. VI.
- Exercises.
  - *Prob.* To draw a parallel through a given point.
  - *Prob.* To draw a tangent to a circle from a point without.
  - *Prob.* To construct a segment on a given line which shall contain a given angle.

SECTION VII.

OF THE ANGLES OF POLYGONS, AND THE RELATION BETWEEN THE ANGLES AND SIDES.

OF TRIANGLES.

PROPOSITION I.

219. *Theorem.*—*The sum of the three angles of a triangle is two right angles.*

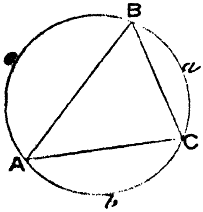


FIG. 168.

DEM.—Conceive a circumference passed through the vertices of the triangle, as *abc*, through the vertices of the triangle **A B C** (*58*). The angle **A** is measured by ½ *arc a*, **B** by ½ *b*, and **C** by ½ *c*. Hence, **A** + **B** + **C** is measured by ½ (*a* + *b* + *c*), or a semi-circumference, and is equal to two right angles (*203*). Q. E. D.

220. COR. 1.—*A triangle can have only one right angle, or one obtuse angle.* Why?

221. COR. 2.—*Two angles of a triangle, or their sum, being given, the third may be found by subtracting this sum from two right angles,* i. e., *either angle is the supplement of the other two.*

222. COR. 3.—*The sum of the two acute angles of a right-angled triangle is equal to one right angle;* i.e., *they are complements of each other.*

223. COR. 4.—*If the angles of a triangle are equal each to each, any one is one-third of two right angles, or two-thirds of one right angle.*

PROPOSITION II.

224. *Theorem.*—*The sides of a triangle sustain the same* GENERAL *relation to each other as their opposite angles; that is, the greatest side is opposite the greatest angle, the second greatest side opposite the second greatest angle, and the least side opposite the least angle.*

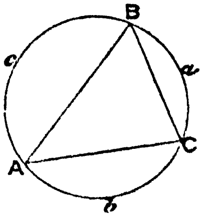


FIG. 169.

DEM.—In the triangle ABC let $C > B > A$ be the order of the values of the angles; then $AB > AC > BC$ is the order of the values of the sides.

For, circumscribe the circumference *abc*. The angle C being greater than B, the arc *c*, the half of which measures C, is greater than the arc *b*, the half of which measures B. Now, the greater arc has the greater chord (*166*). Hence, $AB > AC$. In like manner, if $B > A$, arc $b >$ arc a, and $AC > BC$. If either angle, as C, is obtuse, AB is greater than AC or BC, because it lies nearer the centre (*167*).

225. COR. 1.—Conversely, *The order of the magnitudes of the sides being* $AB > AC > BC$, *the order of the magnitudes of the angles is* $C > B > A$.

[Let the student give the demonstration in form.]

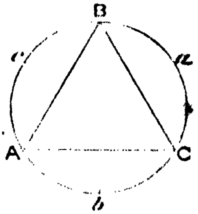


FIG. 170.

226. COR. 2.—*An equiangular triangle is also equilateral; and, conversely, an equilateral triangle is equiangular.*

DEM.—If $A = B = C$, arc $a =$ arc $b =$ arc c, and, consequently, chord $BC =$ chord $AC =$ chord AB. Conversely, if the chords are equal, the arcs are, and hence the angles subtended by these arcs.

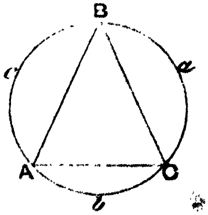


FIG. 171.

227. COR. 3.—*In an isosceles triangle the angles opposite the equal sides are equal; and, conversely, if two angles of a triangle are equal, the sides opposite are equal, and the triangle is isosceles.*

DEM.—If $AB = BC$, arc $a =$ arc c; and hence, angle A, measured by $\frac{1}{2}a$, $=$ angle C, measured by $\frac{1}{2}c$. Conversely, if $A = C$, arc $a =$ arc c; and hence chord $BC =$ chord AB.

228. SCH.—It should be observed that the proposition gives only the *general* relation between the angles and sides of a triangle. It is not meant that the sides are *in the same ratio* as their opposite angles: this is not true. Thus in *Fig.* 172 angle *c* is twice as great as angle *a*; but side *c* is not *twice* as great as side *a*, although it *is greater*. Trigonometry discovers the *exact* relation which exists between the sides and angles.

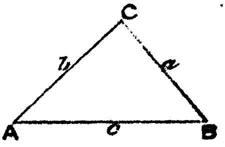


FIG. 172.

PROPOSITION III.

Theorem.—*If from any point within a triangle lines be drawn to the extremities of any side, the included angle is greater than the angle of the triangle opposite this side.*

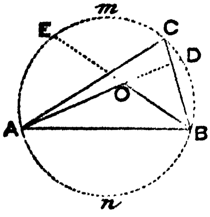


FIG. 173.

DEM.—Let **OA** and **OB** be two lines drawn from any point **O** within the triangle **ABC**, to the extremities of the side **AB**; then angle **AOB** > **ACB**.

For, circumscribe a circle about the triangle. Now, **ACB** is measured by ½ **A*n*B**, but **AOB** is measured by ½ (**A*n*B** + **E*m*D**). Therefore, **AOB** > **ACB**. Q. E. D.

230. An Exterior Angle of a polygon is an angle formed by any side with its adjacent side produced, as **CBD**, *Fig.* 174.

PROPOSITION IV.

231. Theorem.—*An exterior angle of a triangle is equal to the sum of the two interior non-adjacent angles.*

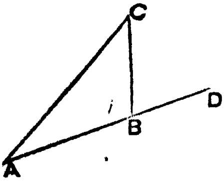


FIG. 174.

DEM.—Let **ABC** be any triangle, and **CBD** an exterior angle; then **CBD** = **A** + **C**.

For **CBD** is the supplement of **CBA** by (***131***), and **CBA** is the supplement of **A** + **C** by (***221***). Hence, **CBD** = **A** + **C**. Q. E. D.

232. COR.—*Either angle of a triangle not adjacent to a specified exterior angle, is equal to the difference of this exterior angle and the other non-adjacent angle.*

Thus, since **CBD** = **A** + **C**, by transposition, **CBD** − **A** = **C**, and **CBD** − **C**

OF QUADRILATERALS.

PROPOSITION V.

233. Theorem.—*The sum of the angles of a quadrilateral is four right angles.*

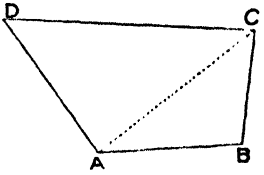


FIG. 175.

DEM.—Let ABCD be any quadrilateral; then DAB + ABC + BCD + CDA = four right angles.

For, draw either diagonal, as AC, dividing the quadrilateral into two triangles. Then, as the sum of the angles of the two triangles is the same as the sum of the angles of the quadrilateral, and the sum of the angles of the triangles is twice two right angles (*219*); the sum of the angles of the quadrilateral is four right angles. Q. E. D.

PROPOSITION VI.

234. Theorem.—*The opposite angles of any quadrilateral which can be inscribed in a circle are supplementary.*

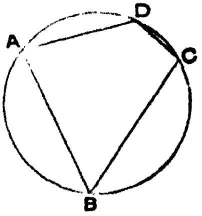


FIG. 176.

DEM.—Let ABCD be any inscribed quadrilateral; then A + C = two *right angles*, and D + B = two *right angles.*

For, A is measured by ½ *arc* BCD, and C is measured by ½ *arc* DAB (*210*). Hence, A + C is measured by one-half a circumference, and is, therefore, equal to two right angles (*203*). In like manner D is measured by ½ *arc* ABC, and B by ½ *arc* ADC. Consequently, D + B is measured by one-half a circumference, and is, therefore, equal to two right angles.

PROPOSITION VII.

235. Theorem.—*The opposite angles of a parallelogram are equal, and the adjacent angles are supplementary.*

DEM.—ABCD, *Fig.* 177, being any parallelogram, A = C, B = D, and B + C, C + D, D + A, and A + B, each = *two right* angles.

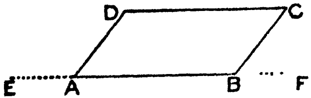


FIG. 177.

For, produce any side, as **AB**, forming the two exterior angles **EAD** and **CBF**. Since **CB** and **DA** are parallel, and **FE** cuts them, the corresponding angles, **CBF** and **DAB** are equal (*152*). Again, since **EF** and **DC** are parallel, and **CB** cuts them, the alternate interior[*] angles **CBF** and **C** are equal (*152*). Hence, as **DAB** and **C** are each equal to **CBF**, they are equal to each other. In a similar manner **D** can be proved equal to **CBA**. [Let the student give the proof.]

That the angles **B** and **C** of the parallelogram are supplemental is evident from (*150*), which proves that the sum of two interior angles on the same side of a secant cutting two parallels is two right angles. For a like reason **A** + **D** = *two right angles*, etc.

FIG. 178.

236. COR. 1.—*The two angles of a trapezoid adjacent to either one of the two sides not parallel are supplemental.*

[Let the student show why.]

237. COR. 2.—*If one angle of a parallelogram is right, the others are also, and the figure is a rectangle.*

PROPOSITION VIII.

238. ***Theorem.***—Conversely to the last, *If the adjacent angles of a quadrilateral are supplementary, or the opposite angles equal, the figure is a parallelogram.*

DEM.—If **A** + **D** = two right angles, **AB** and **DC** are parallel by (*147*). For a like reason, if **D** + **C** = two right angles, **DA** and **CB** are parallel. Again, if **A** = **C** and **D** = **B**, by adding we have **A** + **D** = **C** + **B**. But **A** + **D** + **C** + **B** = four right angles (*233*). Hence, **A** + **D** = two right angles, and **AB** and **CD** are parallel. So, also, **A** + **B** can be shown to be equal to two right angles; and, consequently, **AD** and **CB** are parallel.

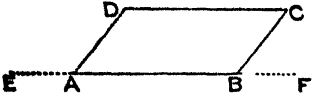


FIG. 179.

---

[*] Interior with reference to the parallels (146).

PROPOSITION IX.

239. *Theorem.*—*If two opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*

DEM.—In (*a*) let **DC** be equal and parallel to **AB**; then is **ABCD** a parallelogram.

For, drawing the diagonal **AC**, it makes the angles **ACD** and **CAB** equal, since they are alternate interior angles (***152***). Conceive the quadrilateral divided in this diagonal into two triangles, as in (*b*). Reverse the triangle **ACB** and place it as in (*c*). Draw **DB**. Since angle **DCA** = angle **CAB**, and **DC** = **BA**, if **CBA** be revolved upon **AC**, **AB** will take the direction **CD**, **B** will fall in **D**, and **CBA** will coincide with **ADC**. Hence, angle **ACB** = angle **DAC**. But in (*a*) these are alternate interior angles made by **AC** with **AD** and **BC**. Therefore, **AD** is parallel to **BC** (***149***). Q. E. D.

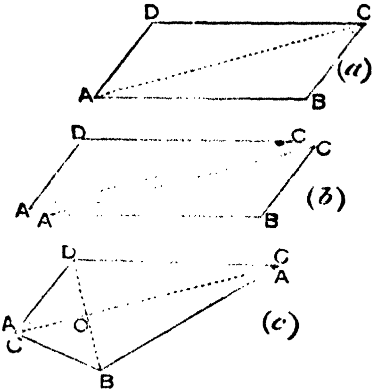


FIG. 180

PROPOSITION X.

240. *Theorem.*—*If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

DEM.—In (*a*) let **AB** = **DC**, and **AD** = **BC**; then is **ABCD** a parallelogram.

For, divide the quadrilateral in the diagonal **AC**, and reversing the triangle **ABC**, place it as in (*c*), and draw **DB**. Since **AB** = **CD**, and **CB** = **AD**, **DB** is perpendicular to **CA** (***130***). Now, revolving **ABC** upon **CA**, it will coincide with **ADC**. Hence, angle **DCA** = angle **CAB**, and **AB** is parallel to **DC**. Also, angle **DAC** = angle **BCA**, and **AD** is parallel to **BC**. Therefore, **ABCD** is a parallelogram. Q. E. D.

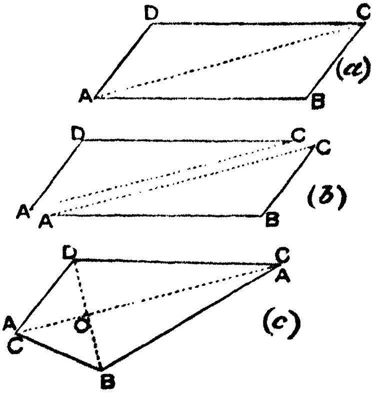


FIG. 181.

PROPOSITION XI.

241. ***Theorem.***—Conversely to the last, *The opposite sides of a parallelogram are equal.*

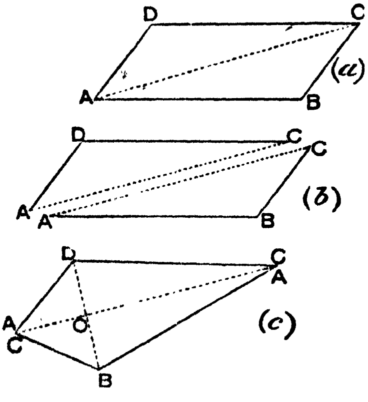


FIG. 182.

DEM.—Let ABCD be a parallelogram; then AB = DC, and AD = CB.

Since DC is parallel to AB, angle DCA = angle CAB. Also, since AD is parallel to BC, angle DAC = angle ACB (*152*). Now, divide the parallelogram (*a*) in the diagonal, and place ABC as in (*c*). Revolve ABC on AC, until it falls in the plane on the other side of AC. Since angle BAC = angle ACD, AB will take the direction CD, and B will fall in CD, or CD produced. Since angle BCA = angle DAC, CB will take the direction AD, and B will fall in AD, or in AD produced. Therefore, as B falls at the same time in AD and CD, it falls at the intersection D, and the triangles coincide. Hence, AB = CD, and AD = CB. Q. E. D.

242. COR. 1.—*Parallels intercepted between parallels are equal.*

243. COR. 2.—*A diagonal of a parallelogram divides it into two equal triangles.*

PROPOSITION XII.

244. ***Theorem.***—*The diagonals of a parallelogram mutually bisect each other.*

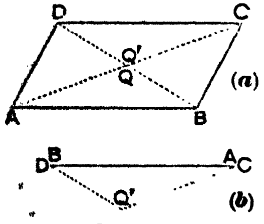


FIG. 183.

M.—Let AC and DB be the diagonals of the parallelogram ABCD (*a*), and Q their intersection; then, DQ = QB, and AQ = QC.

For, take the triangle AQB, and apply it to DQ′C, by placing BA in its equal DC, B falling at D, and A at C, with the vertices Q and Q′ on the same side of this common line, as in (*b*). Now, since angle QBA = Q′DC (*152*), BQ will take the direction DQ′, and Q will fall in DQ′, or in DQ′ produced. For a like reason AQ will take the direction CQ′, and Q will fall in CQ′, or in CQ′ produced. Hence, as Q falls at the same time in DQ′ and CQ′, it falls at their intersection Q′; whence BQ = DQ′, and AQ = CQ′. Q. E. D.

PROPOSITION XIII.

245. Theorem.—*The diagonals of a rhombus bisect each other at right angles.*

Dem.—Let AC and DB be the diagonals of the rhombus ABCD; then are they at right angles to each other, and bisected at Q.

For, since AB = AD, and DC = CB, AC has two of its points equally distant from D and B, and is, therefore, perpendicular to DB, at its middle point (*130*). In like manner D and B are each equidistant from A and C, whence Q is the middle point of AC.

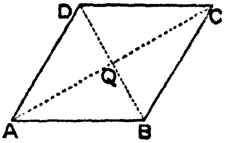


Fig. 184.

246. Cor.—*The diagonals of a rhombus bisect its angles.*

For, revolve ABC upon AC as an axis, and it will coincide with ADC. Hence angles A and C are bisected. In like manner revolve DAB upon DB, and it will coincide with DCB. Hence D and B are bisected.

PROPOSITION XIV.

247. Theorem.—*The diagonals of a rectangle are equal.*

Dem.—Let AC and DB be the diagonals of the rectangle ABCD; then AC = DB.

For, upon AC as a diameter describe a circle. Since D and B are right angles, they are inscribed in semicircles (*211*), and DB is a diameter. Therefore, AC = DB. Q. E. D.

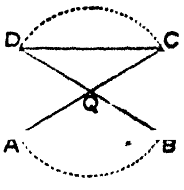


Fig. 185.

248. Cor.—Conversely, *If the diagonals of a parallelogram are equal, the figure is a rectangle.*

Dem.—Since the diagonals of a parallelogram bisect each other, if they are equal, a circumference described from their intersection as a centre, with a radius equal to half of a diagonal, will pass through the vertices of the parallelogram. Hence the diagonals will be diameters, and the angles will be inscribed in semicircles, and consequently will be right angles.

OF POLYGONS OF MORE THAN FOUR SIDES.

249. ***A Salient Angle*** of a polygon is one whose sides, when produced, can only extend *without* the polygon.

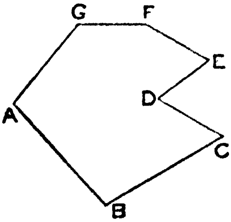


FIG. 186.

250. ***A Re-entrant Angle*** of a polygon is one whose sides, when produced, can extend *within* the polygon.

ILL.—In the polygon ABCDEFG, all the angles are salient except D, which is re-entrant.

1. ***A Convex Polygon*** is a polygon which has only salient angles. A polygon is always supposed to be convex, unless the contrary is stated.

252. ***A Concave or Re-entrant Polygon*** is a polygon with at least one re-entrant angle.

PROPOSITION XV.

253. ***Theorem.***—*The sum of the interior angles of a polygon is equal to twice as many right angles as the polygon has sides, less four right angles.*

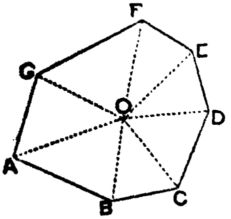


FIG. 187.

DEM.—Let n be the number of sides of any polygon: then the sum of its angles is

n times *two right angles* — 4 *right angles.*

For, from any point O, within, draw lines to the vertices of the angles. As many triangles will thus be formed as the polygon has sides, that is, n. The sum of the angles of these triangles is

n times *two right angles* (**219**).

But this exceeds the sum of the angles of the by the sum of the angles at the common vertex O, that is, by 4 *right* Hence the sum of the angles of the polygon is

n times *two right angles* — 4 *right angles.* Q. E. D.

1. SCH. 1.—The sum of the angles of a pentagon is 5 times *two right angles* — 4 *right angles*, or 6 *right angles.* The sum of the angles of a hexagon is 8 *right angles;* of a heptagon, 10; of an octagon, 12, etc.

255. Sch. 2.—This proposition is equally applicable to triangles and to quadrilaterals. Thus the sum of the angles of a triangle is 3 times *two right angles* — 4 *right angles* (or 6 — 4) = 2 right angles. So also the sum of the angles of a quadrilateral is 4 times *two right angles* — 4 *right angles*, or 4 right angles.

256. Sch. 3.—To find the value of an angle of an equiangular polygon, that is, one whose angles are equal each to each, divide the sum of all the angles by the number of angles.

PROPOSITION XVI.

257. *Theorem.*—*If the sides of a polygon be produced so as to form one exterior angle (and only one) at each vertex, the sum of these exterior angles is four right angles.*

Dem.—Let n be the number of sides of any polygon. At each of the n angles, there is an interior and an exterior angle, whose sum, as A + a, is two right angles. Hence the sum of all the exterior and interior angles is n times *two right angles*. Now, from this sum subtracting the sum of the exterior angles, the remainder is the sum of the interior angles. But, by the preceding proposition, 4 *right angles* subtracted from n times *two right angles*, leaves the sum of the interior angles. Therefore the sum of the exterior angles is 4 *right angles*. Q. E. D.

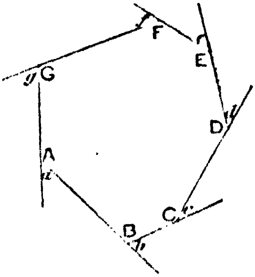


Fig. 188.

OF REGULAR POLYGONS.

PROPOSITION XVII.

258. *Theorem.*—*The angles of an inscribed equilateral polygon are equal; and the polygon is regular.*

Dem.—Let ABCDEF be an inscribed polygon, with AB = BC = CD, etc.; then is angle A = B = C = D, etc., and the polygon is regular.

For, from the centre of the circle draw OF, OA, and OB, and also the perpendiculars Oa and Ob. Revolve OFA upon OA as an axis, until it falls in the plane of **OAB**. Since the chords **FA** and **AB** are equal, the *arc* **FA** = *arc* **AB**, and **F** falls at **B**. Hence the triangles **OFA** and **OAB** coincide. The angle **A** of the polygon is therefore bisected by **OA**; that is, **OAF** = **OAB**. In the same manner **OBA** can be shown equal to **OBC**. Moreover, since **OA** and **OB** are equal oblique lines drawn from a point in the perpendicular, *angle* **OAB** = *angle* **OBA**. Hence, as the half of **A** equals the half of **B**, **A** = **B**. In like manner, **B** can be shown equal to **C**, **C** to **D**, **D** to **E**, etc. Therefore the polygon is equiangular, as well as equilateral, and consequently regular (***117***). Q. E. D.

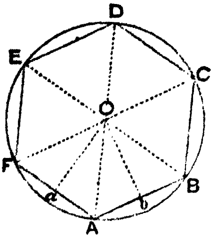


FIG. 189.

PROPOSITION XVIII.

259. Theorem.—*The sides of an inscribed equiangular polygon are equal when their number is* ODD; *and the polygon is regular.*

DEM.—Let **ABCDEFG** be an inscribed equiangular polygon of an *odd* number of sides; then is side **AB** = **BC** = **CD**, etc., and the polygon is regular.

For, from the centre of the circle draw the radii **OA**, **OB**, etc., to the vertices of the polygon, and **O***a*, **O***b*, etc., perpendicular to the sides. Revolve the quadrilateral **OCA***a*, upon **O***a* as an axis until it falls in the plane of **OCB***a*. Since **O***a* is perpendicular to the chord **AB**, **A***a* = *a***B**, and **A** will fall at **B**. Also, as the angle **A** of the polygon = **B**, **AC** will fall in **BC**. Now **C** falls at the same time in the arc **BCD** (***158***) and in **BC**, and hence falls at their intersection **C**. Therefore **AC** = **BC**. In like manner revolving **OBC***c* upon **O***c* as an axis, **BC** is found equal to **ED**. So also we can show that **ED** = **FC**; then that **FC** = **AB**; then that **AB** = **DC**; and finally, that **DC** = **EF**. Hence we have **CA** = **BC** = **ED** = **FC** = **AB** = **DC** = **EF**; and as the polygon is equiangular by hypothesis, it is regular (***117***). Q. E. D.

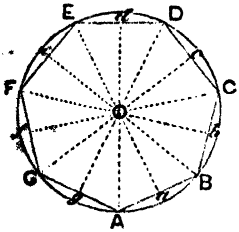


FIG. 190.

260. SCH.—It is easy to see that the above argument would fail in the case of a polygon of an even number of sides, because, in going around the second time the *same* sides would coincide as in going around the first time. Moreover, we can readily inscribe an equiangular polygon of an *even* number of sides which shall not be regular.

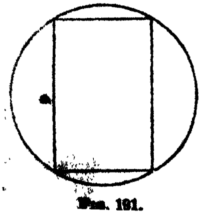


FIG. 191.

PROPOSITION XIX.

261. Theorem.—*The sides of a circumscribed equiangular polygon are equal; and the polygon is regular.*

Dem.—Let ABCDEF be a circumscribed polygon, with angle A = B = C, etc.; then is AB = BC = CD, etc., and the polygon is regular.

For, from the centre of the circle, draw OA, OB, etc., to the vertices of the polygon, and O*a*, O*b*, etc., to the points of tangency. The latter will be perpendicular to the sides by (*173*). Now reverse the triangle A*a*O, and apply it to A*b*O, placing O*a* in its equal O*b*; *a*A will take the direction *b*A. Then will OA of the triangle A*a*O, fall in OA of the triangle A*b*O, since there cannot be two equal oblique lines on the same side of O*b* (*140*). Hence *angle b*AO = *angle a*AO, and *b*A = *a*A. In the same way it can be shown that OB, OF, etc., bisect the other angles, and that *b*B = B*c*, etc. Whence, as the polygon is equiangular, these halves are equal, that is, OA*a* = OF*a*, etc. Then, as OA and OF make equal angles with AF, they cut off equal distances from *a*, and A*a* = *a*F. So, likewise, we can show that A*b* = *b*B, and that each side is bisected at the point of tangency. Therefore, as the halves of the sides are equal, the polygon is equilateral, as well as equiangular, and consequently regular (*117*). Q. E. D.

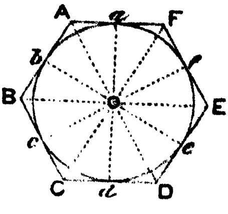


Fig. 192.

PROPOSITION XX.

262. Theorem.—*The angles of a circumscribed equilateral polygon are equal when their number is* ODD; *and the polygon is regular.*

Dem.—Let ABCDE be a circumscribed polygon with AB = BC = CD, etc.; then is *angle* A = B = C = D, etc., and the polygon is regular.

In the same manner as in the preceding demonstration, we may show that OA, OB, etc., bisect the angles of the polygon. [The student should go through the process.] Then revolve the triangle AOE upon AO as an axis till it falls in the plane of AOB; and as angle OAE = angle OAB, and AE = AB, the triangles will coincide. Hence angle OEA, the half of angle E of the polygon, equals angle OBA the half of B, and E = B. In like manner revolving AOB upon OB, we can show that A = C. So also we find B = D, and D = A. Therefore the polygon is equiangular as well as equilateral, and consequently regular. Q. E. D.

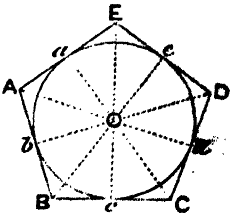


Fig. 193.

263. Sch.—That the above style of argument fails in the case of a polygon of an *even* number of sides, may be observed by attempting to apply it. Thus, from *Fig.* 192, we would have A = C, B = D, C = E, D = F, E = A, and F = B. From these we have A = C = E, and B = D = F. But the process will not give any one of the first three angles equal to any one of the second set. That is, it does not follow that two *adjacent* angles are equal in case the number of sides is *even*. We can readily construct a circumscribed equilateral polygon which shall not be equiangular.

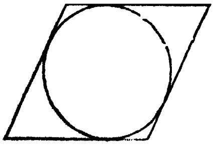


Fig. 194.

PROPOSITION XXI.

264. ***Theorem.***—*A circumference may be circumscribed about any regular polygon.*

Dem.—Let ABCDEF be a regular polygon. Bisect AF with a perpendicular O*a*. Any point in this perpendicular is equidistant from A and F. Bisect AB, adjacent to AF, with a perpendicular, as O*b*. Any point in this perpendicular is equidistant from A and B. Hence the intersection of these perpendiculars, O, is equidistant from A, F, and B, and a circumference described from O as a centre, with a radius OA, will pass through F and B. Now revolve the quadrilateral FO*b*A upon O*b* as an axis until it falls in the plane of CO*b*B, *b*A will fall in its equal *b*B; and since angle A = angle B, and side AF = side BC, F will fall at C. Thus it appears that the circumference described from O, and passing through F, A, and B, also passes through C. In a similar manner it can be shown that the same circumference passes through all the vertices, and hence is circumscribed. Q. E. D.

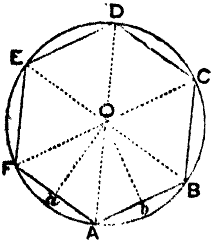


Fig. 195.

265. Cor. 1.—*A circumference may be inscribed in any regular polygon.*

Dem.—For, having circumscribed one about it, the equal sides become equal chords, and hence are equally distant from the centre. If, therefore, a circle be drawn from O as a centre, with O*a* as a radius, it will touch every side of the polygon at its middle point.

6. Cor. 2.—*The centres of the inscribed and circumscribed circles coincide.*

The Centre of a regular polygon is the common centre of its inscribed and circumscribed circles.

268. ***An Angle at the Centre*** of a regular polygon is the angle included by two lines drawn from the centre to the extremities of a side, as FOA, AOB.

269. Cor. 3.—*The angles at the centre of a regular polygon are equal each to each; and any one is equal to four right angles divided by the number of sides of the polygon.*

270. ***The Apothem*** of a regular polygon is the distance from the centre to any side, and is the radius of the inscribed circle.

PROPOSITION XXII.

271. ***Theorem.***—*The side of a regular inscribed hexagon is equal to the radius.*

Dem.—Let ABCDEF be a regular inscribed hexagon; then is any side, as BC, equal to OB, the radius.

In the triangle BOC the angle O is measured by the arc BC, or $\frac{1}{6}$ of a circumference, and hence is $\frac{1}{6}$ of 4 right angles, or $\frac{2}{3}$ of a right angle. Angle ABC is measured by $\frac{1}{2}$ arc CDEFA, or $\frac{1}{3}$ of a circumference. Hence angle OBC, which is $\frac{1}{2}$ of ABC, is measured by $\frac{1}{2}$ of $\frac{1}{3}$, or $\frac{1}{6}$ of a circumference, and is, consequently, equal to BOC. So also OCB, the half of DCB, is measured by $\frac{1}{6}$ of a circumference. Hence OCB is equiangular, and consequently equilateral ∴ BC = OB. Q. E. D.

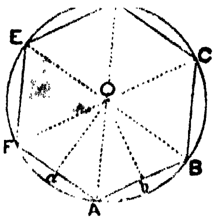


Fig. 196.

272. A *Broken Line* is said to be *Convex* when no one of its parts will, when produced, enter the space included between it and a line joining its extremities.

PROPOSITION XXIII.

273. ***Theorem.***—*A Convex broken line is less than any broken line which envelops it and has the same extremities.*

Dem.—Let *Abcd*B be a broken line enveloped by the broken line ACDEFB, and having the same extremities A and B; then is *Abcd*B < ACDEFB.

For, produce the parts of *Abcd*B till they meet the enveloping line, as A*b* to *e*, *bc* to *f*, and *cd* to *g*. Now, since a straight line is the shortest path between two points, A*e* < AC*e*, *bf* < *be*DE*f*,

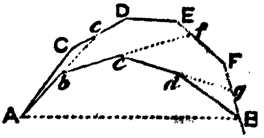


Fig. 197.

$eg < efFg$, and $dB < dgB$. Hence, if a point starts from A to move to B, AeDEFB will be a shorter path than ACDEFB, AbfFB shorter than AeDEFB, AbcgB shorter than AbfFB, and AbcdB shorter than AbcgB. Therefore, AbcdB is shorter than ACDEFB. Q. E. D.

274. COR. 1.—*The sum of any two sides of a triangle is greater than the third side.*

This is the same as the axiom that the shortest distance between two points is a straight line.

275. COR. 2.—*The difference between any two sides of a triangle is less than the third side.*

DEM.—Let a, b, and c be the sides. By Corollary 1st, $a + b > c$. Therefore, transposing, $a > c - b$.

276. COR. 3.—*If from any point within a triangle lines be drawn to the extremities of any side, the sum of these lines is less than the sum of the other two sides of the triangle.*

EXERCISES.

1. Given two angles of a triangle, to find the third.

SUG'S.—The student should draw two angles on the blackboard, as a and b, and then proceed to find the third. The figure will suggest the method. The third angle is c.

The solution is effected also by constructing the two given angles at the extremities of any line, and producing the sides till they meet.

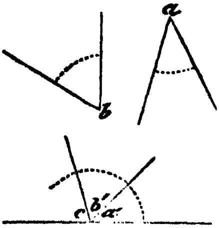


FIG. 198.

2. Two angles of a triangle are respectively ⅘ and ⅓ of a right angle. What is the third angle?

3. The angles of a triangle are respectively ⅔, ½, and ⅚ of a right angle. Which is the greatest side? Which the least? Can you tell the ratio of the sides?

4. What is the value of one of the equal angles of an isosceles triangle whose third angle is ⅓ of a right angle?

5. Two consecutive angles of a quadrilateral are respectively ⅘ and ⅔ of a right angle, and the other two angles are mutually equal to

each other. What is the form of the quadrilateral? What the value of each of the two latter angles?

6. One of the angles of a parallelogram is $\frac{5}{3}$ of a right angle. What are the values of the other angles?

7. The two opposite angles of a quadrilateral are respectively $\frac{6}{5}$ and $\frac{4}{5}$ of a right angle. Can a circumference be circumscribed? If so, do it.

8. Two of the opposite sides of a quadrilateral are parallel, and each is 15 in length. What is the figure? Do these facts determine the angles?

9. Two of the opposite sides of a quadrilateral are 12 each, and the other two 7 each. What do these facts determine with reference to the form of the figure?

10. What is the value of an angle of a regular dodecagon?

11. What is the sum of the angles of a nonagon? What is the value of one angle of a regular nonagon? Of one exterior angle?

12. What is the regular polygon, one of whose angles is $1\frac{3}{7}$ right angles?

13. What is the regular polygon, one of whose exterior angles is $\frac{2}{3}$ of a right angle?

14. Can you cover a plane surface with equilateral triangles without overlapping them or leaving vacant spaces? With quadrilaterals? Of what form? With pentagons? Why? With hexagons? Why? What insect puts the latter fact to practical use? Can you cover a plane surface thus with regular polygons of more than 6 sides? Why?

15. Is an equilateral hexagon circumscribed about a circle necessarily regular? A heptagon? An octagon? A nonagon?

16. Is an equiangular circumscribed quadrilateral necessarily regular? A pentagon? A hexagon? A heptagon?

17. Is an equilateral inscribed pentagon necessarily regular? An octagon? How is it if they are equiangular; are they necessarily equilateral and regular?

SYNOPSIS.

- **ANGLES AND SIDES OF POLYGONS.**
  - **TRIANGLES.**
    - **PROP.** I. Sum of angles.
      - *Cor.* 1. Only one right or obtuse.
      - *Cor.* 2. Two angles given.
      - *Cor.* 3. Acute angles if right angled
      - *Cor.* 4. One angle if equiangular.
    - **PROP.** II. Sides and opp. angles.
      - *Cor.* 1. Converse.
      - *Cor.* 2. Equiangular, equilateral, and converse.
      - *Cor.* 3. Isosceles, equiangular, and converse.
      - *Sch.* These only general relations.
    - PROP. III. Angle within a triangle.
    - DEF. Exterior angle.
    - PROP. IV. Exterior angle.—*Cor.* Non-adjacent interior.
  - **QUADRILATERALS.**
    - PROP. V. Sum of angles.
    - PROP. VI. Angles of inscribed.
    - **PARALLELOGRAMS.**
      - PROP. VII. Angles of.
        - *Cor.* 1. Of a trapezoid.
        - *Cor.* 2. Of a rectangle.
      - PROP. VIII. Converse to last.
      - PROP. IX. Two op. sides of a quadrilat'l equal and parallel.
      - PROP. X. Opposite sides of a quadrilateral equal.
      - PROP. XI. Converse to last.
        - *Cor.* 1. Parallels intercepted bet. parallels.
        - *Cor.* 2. Diagonal of a parallelogram.
      - DIAGONALS.
        - PROP. XII. Bisect.
        - PROP. XIII. Of a rhombus.—*Cor.* Bisect angles.
        - PROP. XIV. Of a rectangle.—*Cor.* Converse.
  - **POLYGONS OF MORE THAN 4 SIDES.**
    - DEF'S.—Salient angle.—Re-entrant.—Convex polygon.—Concave.
    - PROP. XV. Sum of angles.
      - *Sch.* 1. Application.
      - *Sch.* 2. Applied to triangles.
      - *Sch.* 3. Angle of equiangular polygon.
    - PROP. XVI. Sum of exterior angles.
    - REGULAR.
      - PROP. XVII. Equilateral inscribed, regular.
      - PROP. XVIII. Equiangular inscribed if odd No. of sides.
        - *Sch.* Fails for even No.
      - PROP. XIX. Equiangular circumscribed, regular.
      - PROP. XX. Equilateral circbd. if odd No. of sides.
        - *Sch.* Fails for even No.
      - PROP. XXI. Circf. can be circumscribed.
        - *Cor.* 1. Inscribed.
        - *Cor.* 2. Centres.
        - *Def.* Angle at cntr.
        - *Cor.* 3. Value of angle at centre.
        - *Def.* Apothem.
      - PROP. XXII. Side of inscribed hexagon.
  - **DEF.** Convex Broken Line.
  - **PROP.** XXIII. Convex broken line < than —.
    - *Cor.* 1. Sum of two sides of triangle.
    - *Cor.* 2. Diff. of two sides of triangle.
    - *Cor.* 3. Lines from point within triangle.
  - **EXERCISES.**

SECTION VIII.

OF EQUALITY.

277. ***Equality*** signifies likeness in every respect.

278. The equality of magnitudes is usually shown by applying one to the other, and observing that they coincide.

PROPOSITION I.

279. ***Theorem.***—*Two straight lines of the same length are equal magnitudes.**

DEM.—Let AB and CD be two straight lines of the same length; then are they equal.

For, conceive the extremity C of CD placed at A, and the other extremity somewhere in AB, or in AB produced, as the case may be. Now, the point which traces AB passes through all points in the direction of B from A; and hence, if CD is traced from A towards B, it will pass through the same points as far as they mutually extend. The lines therefore coincide, as far as they both extend; and, being of the same length, D falls at B, and they coincide throughout; they are, therefore, equal. Q. E. D.

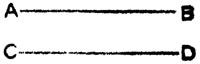


FIG. 189.*

ILL.—The truth of this theorem is so evident, that the student may fail to see the point of the demonstration. Let him see if he can say the same things of two curved lines AmB, and CnD, which are of the same length.

The substance of the demonstration is as follows: A line has two properties, and *only* two, *form* and *magnitude*. *Straight* lines, being of the same form, if they are of the same magnitude, are alike in all respects; *i. e.*, they are equal. Now, a line, as a magnitude, has only one dimension, viz., length. If, therefore, two lines have the same length, they have the same magnitude.

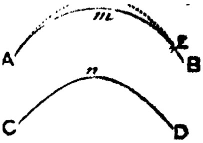


FIG. 190.

---

* See Preface.

PROPOSITION II.

280. *Theorem.*—*Two circles whose radii are of the same length are equal;* i. e., *the circumferences are equal, and the circles equal.*

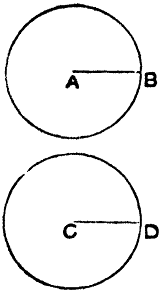


FIG. 200.

DEM.—Let there be two circles whose radii AB and CD are of the same length; then are the circles equal.

For, place the second circle on the first, with the centre C at A, and CD in AB. As CD = AB, D will fall at B. Now, every point in the plane at a distance AB from A is in the circumference of circle A. But every point at a distance CD from the common centre is in the circumference of circle C. Hence, the two figures coincide, and the circles are alike in all respects, *i. e.*, are equal. Q. E. D.

OF ANGLES.

PROPOSITION III.

281. *Theorem.*—*Two angles whose sides are parallel, two and two, and lie in the same or in opposite directions from their vertices, are equal.*

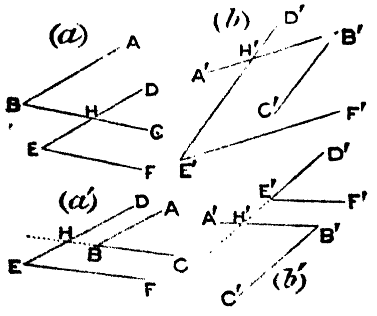


FIG. 201.

DEM.—1st. In (*a*) or (*a'*) let B and E have BA and ED parallel, and extending in the same direction from the vertices, and also BC and EF; then are B and E equal. For, produce (if necessary) either two sides which are not parallel, till they intersect, as at H; then are the corresponding angles DHC and DEF, and DHC and ABC equal (*152*). Hence, ABC = DEF.

2nd. In (*b*) and (*b'*) let B' and E' have B'A' parallel with E'F', but extending in an opposite direction from its vertex; and in like manner B'C' parallel with, but extending in an opposite direction from E'D'; then are B' and E' equal. For, produce (if necessary) two of the sides which are not parallel till they intersect, as at H'; then D'H'B' = the corresponding angle D'E'F', and also = the alternate interior angle A'B'C'; whence A'B'C' = D'E'F'. Q. E. D.

PROPOSITION IV.

282. Theorem.—*If two angles have two sides parallel and extending in the same direction with each other, while the other two sides are parallel and extend in opposite directions from each other, the angles are supplemental.*

DEM.—Let ABC and DEF be two angles, having BC and ED parallel, and extending in the same direction from the vertices, and AB and EF parallel, and extending in opposite directions from the vertices; then are ABC and DEF supplements of each other.

For, produce the two sides not parallel, if necessary, till they meet. Now, BHD is the supplement of BHE by (*131*), BHE = the alternate interior angle DEF, and BHD = the corresponding angle ABC. Therefore, ABC is the supplement of DEF. Q. E. D.

[This demonstration is adapted to the upper cut; let the student adapt it to the lower.]

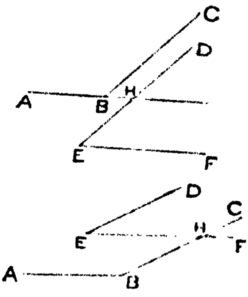


FIG. 202.

PROPOSITION V.

283. Theorem.—*If two angles have their sides respectively perpendicular to each other, the angles are either equal or supplementary.*

DEM.—Let BA be perpendicular to EF or to E'F', and BC to ED; then is ABC = DEF. For, through B draw BO and BN, respectively parallel to ED and EF; then by the preceding propositions NBO = DEF, and is the supplement of F'E'D. But NBA = OBC, since both are right angles. Take away OBA from each, and we have NBO = ABC; and as NBO is the supplement of F'E'D, ABC is also the supplement of F'E'D. Q. E. D.

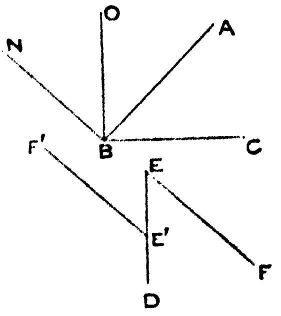


FIG. 203.

OF TRIANGLES.

PROPOSITION VI.

284. Theorem.—*Two triangles which have two sides and the included angle of one equal to two sides and the included angle of the other, each to each, are equal.*

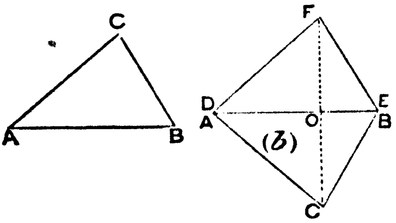


Fig. 204.

Dem.—Let ABC and DEF be two triangles, having AC = DF, AB = DE, and angle A = angle D; then are the triangles equal.

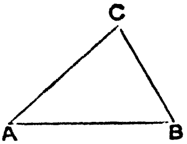
For, place the triangle ABC in the position (*b*), the side AB in its equal DE, and the angle A adjacent to its equal angle D. Then revolving ABC upon DB, until it falls in the plane on the opposite side of DB, since angle A = angle D, AC will take the direction DF; and as AC = DF, C will fall at F. Hence BC will fall in EF, and the triangles will coincide. Therefore the two triangles are equal. Q. E. D.

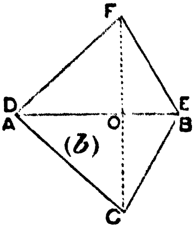
We may also make the application of ABC to DEF directly, as in (*85*). The method here given is used for the purpose of uniformity in this and the following. We may observe that in this, as in the other cases, DB is perpendicular to FC, and bisects it at O. This fact might easily be shown, and the demonstration be based upon it.

285. Sch.—This proposition signifies that the two triangles *are equal in all respects*, *i. e.*, that the two remaining sides are equal, as CB = FE; that angle C = angle F, angle B = angle E, and that the areas are equal.

PROPOSITION VII.

286. Theorem.—*Two triangles which have two angles and the included side of the one equal to two angles and the included side of the other, each to each, are equal.*





Dem.—Let **ABC** and **DEF** be two triangles, having angle **A** = angle **D**, angle **B** = angle **E**, and side **AB** = side **DE**; then are the triangles equal.

For, place **ABC** in the position (*b*), the side **AB** in its equal **DE**, the angle **A** adjacent to its equal angle **D**, and **B** adjacent to its equal angle **E**. Then revolving **ABC** upon **DB** till it falls in the plane on the same side as **DFE**, since angle **A** = angle **D**, **AC** will take the direction **DF**, and **C** will fall somewhere in **DF** or **DF** produced. Also, since angle **B** = angle **E**, **BC** will take the direction **EF**, and **C** will fall somewhere in **EF**, or **EF** produced. Hence, as **C** falls at the same time in **DF** and **EF**, it falls at their intersection **F**. Therefore the two triangles coincide, and are consequently equal. Q. E. D.

287. Cor.—*If one triangle has a side, its opposite angle, and one adjacent angle, equal to the corresponding parts in another triangle, each to each, the triangles are*

For the third angle in each is the supplement of the sum of the given angles, and they are consequently equal. Whence the case is included in the proposition.

288. Sch.—A triangle may have a side and one adjacent angle equal to a side and an adjacent angle in another, and the second adjacent angle of the first equal to the angle *opposite* the equal side in the second, and the triangles *not* be equal. Thus, in the figure, **AB** = **C′A′**, **A** = **A′**, and **B** = **B′**; but the triangles are evidently not equal. [Such triangles are, however, *similar*, as will be shown hereafter.]

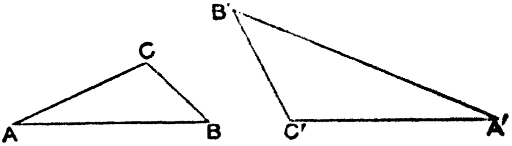


Fig. 200.

PROPOSITION VIII.

289. ***Theorem.***—*Two triangles which have two sides and an angle opposite one of these sides, in the one, equal to the corresponding* *parts in the other, are equal, if of these two sides the one opposite the given angle is equal to or greater than the one adjacent.*

Dem.—In the triangles ABC and DEF, let AC = DF, CB = FE, A = D, and CB (= FE) $\geqq$ AC (= DF); then are the triangles equal.

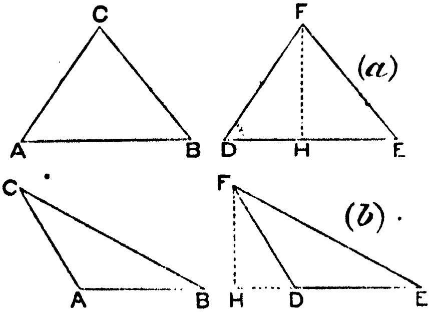


Fig. 207.

For, apply AC to its equal DF, the point A falling at D and C at F. Since A = D, AB will take the direction DE. Let fall the perpendicular FH upon DE, or DE produced. Now, CB being $\geqq$ DF, cannot fall between it and the perpendicular, but must fall in FD or beyond both. As there can be but one line on the same side of the perpendicular equal to CB, and as FE = CB, CB must fall in FE. Hence, the two triangles coincide, and are consequently equal. Q. E. D.

290. Sch. 1.—If A and D are acute and CB (= FE) = AC (= DF), the triangles are isosceles. If A and D are right or obtuse, CB (=FE) must be *greater* than AC (= DF), in order that there may be a triangle, since the right or obtuse angle is the greatest angle in a triangle, and the greatest side is opposite the greatest angle. This impossibility appears also from the demonstration above.

Sch. 2.—If A and D are *acute*, and the side opposite A, *i. e.*, CB, is less than AC, it must be equal to or greater than the perpendicular CI (= FH) in order to have a triangle. Then, applying AC to DF, and observing that AB takes the direction DE, and that EF, which = CB, being intermediate in length between DF and FH, may lie on either side of FH, we see that ABC may or may not coincide with DEF. Whether it does or not will depend upon whether angle C = angle F, or whether AB = DE. This is the AMBIGUOUS CASE in the solution of triangles, and should receive special attention.

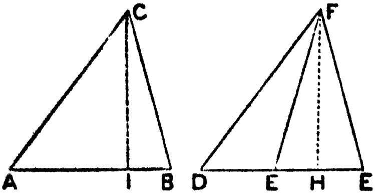


Fig. 208.

PROPOSITION IX.

292. Theorem.—*Two triangles which have the three sides of the one equal to the three sides of the other, each to each, are equal.*

DEM.—Let ABC and DEF be two triangles, in which AB = DE, AC = DF, and BC = EF; then are the triangles equal.

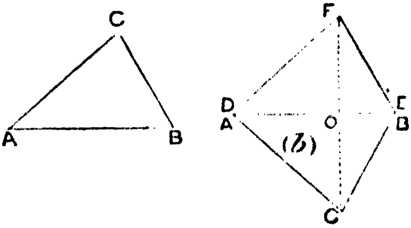


FIG. 209.

For, place the triangle ABC in the position (*b*), and the side AB in its equal DE, so that the other equal sides shall be adjacent, as AC adjacent to DF, and BC to EF. Draw FC. Now, since DC = DF, and EC = EF, DB is perpendicular to FC at its middle point (*130*). Hence, revolving ABC upon DB, it will coincide with DEF when brought into the plane of the latter Therefore the two triangles are equal. Q. E. D.

293. COR.—*In two equal triangles, the equal angles lie opposite the equal sides.*

294. SCH.—If the triangles compared, as in the three preceding propositions, have an obtuse angle, and the two sides first brought together are sides about the obtuse angle, the figure will take the form in the margin; but the demonstration will be the same. When the three sides are the given equal parts, the form of figure given in the demonstration above can always be secured by bringing together the two *greatest* sides.

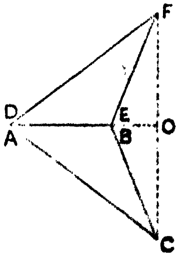


FIG. 210.

PROPOSITION X.

295. Theorem.—*If two triangles have two sides of the one respectively equal to two sides of the other, and the included angles unequal, the third sides are unequal, and the greater third side belongs to the triangle having the greater included*

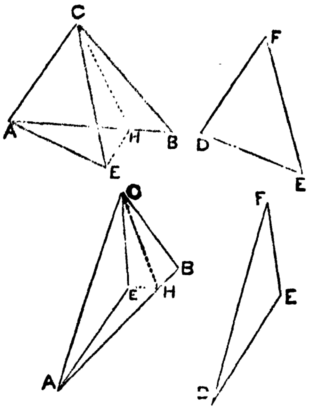


FIG. 211.

Dem.—Let ACB and DEF be two triangles having AC = DF, CB = FE, and C > F; then is AB > DE.

For, placing the side DF in its equal AC, since angle F < angle C, FE will fall within the angle ACB, as in CE. Then let the triangle ACE = the triangle DFE. Bisect ECB with CH, and draw HE. The triangles HCB and HCE have two sides and the included angle of the one, respectively equal to the corresponding parts of the other, whence HE = HB. Now AH + HE > AE; but AH + HE = AH + HB = AB. Therefore, AB > AE. Q. E. D.

296. Cor. — Conversely, *If two sides of one triangle are respectively equal to two sides of another, and the third sides unequal, the angle opposite this third side is the greater in the triangle which has the greater third side.*

Dem.—If AC = DF, CB = FE, and AB > DE, angle C > angle F. For, if C = F, the triangles would be equal, and AB = DE (*284*); and, if C were less than F, AB would be less than DE, by the proposition. But both these conclusions are contrary to the hypothesis. Hence, as C cannot be equal to F, nor less than F, it must be greater.

PROPOSITION XI.

297. ***Theorem.***—*Two right angled triangles which have the hypotenuse and one side of the one equal to the hypotenuse and one side of the other, each to each, are equal.*

Dem.—In the two triangles ABC and DEF, right angled at B and E, let AC = DF, and BC = EF; then are the triangles equal.

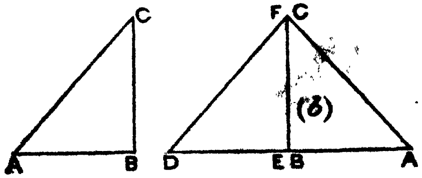


FIG. 212.

For, place BC in its equal EF, so that the right angles shall be adjacent, the angles A and D lying on opposite sides of EF, as in (*b*). Since E and B are right angles, DA is a straight line. Now, since equal oblique lines, as FD and CA, cut off equal distances from the foot of the perpendicular (*141*), DE = BA; and revolving CAB upon FB, the two triangles will coincide when CAB falls in the plane on the side D. Therefore, the triangles are equal. Q. E. D.

PROPOSITION XII.

298. Theorem.—*Two right angled triangles having the hypotenuse and one acute angle of the one equal to the hypotenuse and an acute angle of the other, are equal.*

Dem.—One acute angle in each being equal, the other acute angles are equal, since they are complements of the same angle (*222*). The case is, then, that of two angles (the acute angles in each), and their included side (the hypotenuse), and falls under (*286*).

PROPOSITION XIII.

299. Theorem.—*Two right angled triangles having a side and a corresponding acute angle in each equal, are equal.*

This also falls under (*286*). Let the student show why.

OF QUADRILATERALS.

PROPOSITION XIV.

). Theorem.—*Two quadrilaterals are equal when the following parts are equal, each to each, in both quadrilaterals, and similarly arranged:*

1. *The triangles into which either diagonal divides the quadrilaterals.*
2. *The four sides and either diagonal.*
3. *The four sides and one angle.*
4. *Three sides and the two included angles.*
5. *Three angles and any two sides, if the other two sides are non-parallel.*

Dem.—1. This case is demonstrated by applying one quadrilateral to the other.

2. This case is reduced to the former by (*292*).

3. Drawing the diagonal opposite the known angle, this case is reduced to the first by (*284*), and (*292*).

4. This is demonstrated by applying one quadrilateral to the other; or, drawing a diagonal, it may be reduced to case first by (*284*).

5. In this case the quadrilaterals are mutually equiangular by (*255*). If, then, the two sides are adjacent, by drawing the diagonal joining their extremi-

ties the case is brought under the first by (*284*), and (*286*). If, however, the two known sides are non-adjacent, by producing the unknown sides two triangles are formed in each figure, which are mutually equal by (*286*), and the case comes under the axiom concerning equals subtracted from equals.

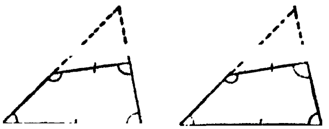


FIG. 213.

[Let the pupil draw the figures and give the demonstrations in full form. Trapeziums should be used; although it should be seen in each case—except the fifth—that the truth applies to any other form of quadrilateral.]

PROPOSITION XV.

301. Theorem.*—Two parallelograms having two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, are equal.*

DEM.—Let AC and EC be two parallelograms, with AD = EH, AB = EF, and A = E; then are they equal.

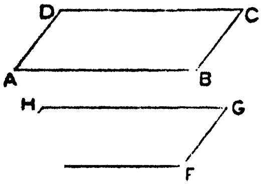


FIG. 214.

For, applying the angle E to A, since EH = AD, H will fall at D; and since EF = AB, F will fall at B. Now, through D but one line can be drawn parallel to AB; hence HG will fall in DC, and G will be found in DC, or in DC produced. In like manner, since but one parallel to AD can be drawn through B, FG must fall in BC, and G be found in BC, or in BC produced. Therefore, as G falls at the same time in DC and BC, it falls at C, and the parallelograms coincide.

302. COR.—*Two rectangles of the same base and altitude are equal.*

OF POLYGONS.

PROPOSITION XVI.

303. Theorem.*—Two polygons of the same number of sides, having all the parts of the one except three angles, known to be respectively equal to the corresponding parts of the other, are equal.*

DEM.—If the two polygons AE* and A'E', have all the parts of the one equal to the corresponding parts of the other, each to each, except three angles; then are the polygons equal.

---

* It is often more convenient to read a polygon by two letters, instead of all those at the

1st. When the three unknown angles are consecutive, as C, F, E, and C', F', E'. Draw CE, and C'E'. Apply polygon A'E' to AE, beginning with g' in its equal g: A' = A, and $a' = a$; hence, B' falls at B; B' = B, and $b' = b$, hence, C' falls at C; etc. Thus, we may show that the perimeters coincide till we reach E' and E. Then will C'E' = CE, and the triangles CFE and C'F'E', having their corresponding sides respectively equal, are themselves equal, and the polygons coincide throughout.

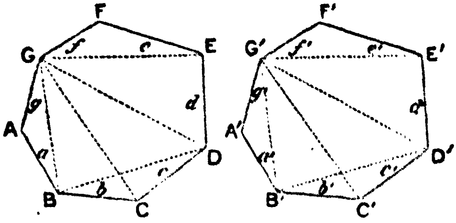


Fig. 215.

2d. When two of the unknown angles are consecutive, and the third not consecutive with these, as C, E, D, and C', E', D'. From the angle which is not consecutive with the other two, draw diagonals to the other angles, as CE, CD, and C'E', C'D'. Now, C'A'B'C'D' can be applied to CABCD, and C'F'E' to CFE, in the ordinary way. Hence, the triangles C'E'D' and CED are mutually equilateral, and consequently equal. Therefore the polygons are equal.

3d. When no two of the three unknown angles are consecutive, as C, B, D, and C', B', D'. Join the unknown angles by diagonals, as CB, CD, BD, and C'B', C'D', B'D'. Now, polygon C'F'E'D' can be applied to CFED, D'C'B' to DCB, and C'A'B' to CAB in the ordinary way. Hence, the triangles C'D'B' and CDB are mutually equilateral, and consequently equal. Therefore the polygons are equal.*

304. Cor.—*Two quadrilaterals having their corresponding sides equal, and an angle in one equal to the corresponding angle in the other, are equal.*

PROPOSITION XVII.

305. Theorem.—*Two polygons of the same number of having all the parts of the one except two angles and the included side, known to be respectively equal to the corresponding parts of the other, are equal.*

I.—Let the unknown parts be C, c, D, and C', c', D'. From any other two of the mutually equal angles, as G and G', draw the diagonals GC, GD, G'C', G'D',

---

* Notice that in each case the unknown angles are to form the vertices of triangles, which the argument shows to be equilateral, and therefore equal. In Case 1st, we have to draw only one line in order to give the triangles, as two sides are sides of the polygon; in Case 2d, we have to draw two sides, and in Case 3d, three sides, for analogous reasons.

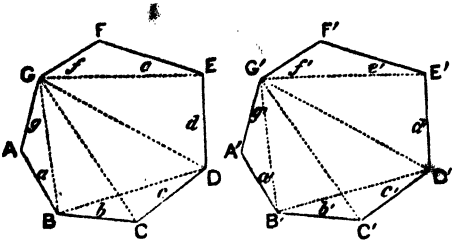


FIG. 216.

to the unknown angles. Then the polygon G′F′E′D′ can be applied in the ordinary way to GFED, *f′* being placed in *f*, etc. So also G′A′B′C′ can be applied to GABC, beginning with *g′* in its equal *g*. Hence, angle F′G′D′ = FGD, A′G′C′ = AGC; and, adding, F′G′D′ + A′G′C′ = FGD + AGC. Subtracting these equals from G′ = G, we have C′G′D′ = CGD. Whence the triangles C′G′D′ and CGD have two sides and their included angle equal in each, and are equal; therefore the polygons are equal in all their parts.

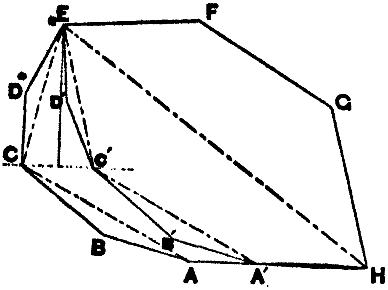


FIG. 217.

306. SCH.—When the unknown angles are both separated from the unknown side, the polygons may or may not be equal—the case is ambiguous. Thus, if C and E are the unknown angles and AH the unknown side, the polygons ABCDEFG, and A′B′C′D′EFG fulfill the conditions, but are not equal. By drawing CE, CA, and EH, the case is reduced to that of two quadrilaterals having all the parts equal, each to each, except two angles and their non-adjacent side; in which case the quadrilaterals are not necessarily equal.

So, also, when one of the unknown angles is adjacent to the unknown side and the other separated, the polygons may or may not be equal. Thus, let

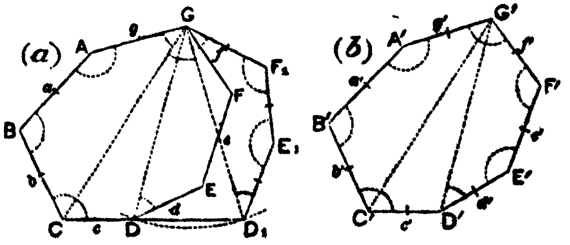


FIG. 218.

the unknown parts be D, *c*, C, and D′, *c′*, C′. From the separated angle draw the diagonals to the extremities of the unknown side, as GC, GD (or GD₁), and G′C′, Y. In the usual way G′A′B′C′ can be applied to GABC, and G′F′E′D′ to

CFED. Whence G'C' = GC, G'D' = GD, and angle G'C'D' = GCD. Thus the case is reduced to that of two triangles having two sides and an angle opposite one of them mutually equal, and is, therefore, ambiguous. The polygon (*a*) may have the part corresponding to C'F'E'D' situated as CFED, or as $CF_1E_1D_1$. In the former case the polygons are equal, in the latter not.

307. Cor.—*Two quadrilaterals having three sides and the corresponding angles included by these sides equal, are equal.*

This falls under the 1st case.

308. Sch.—If the three unknown or excepted parts are all sides, the polygons are not necessarily equal, as will appear by an inspection of the figure. The unmarked sides being the excepted ones, the polygons may be those included by the continuous lines, or those included in part by the broken lines, all the parts being equal in each two, except the three unknown ones.

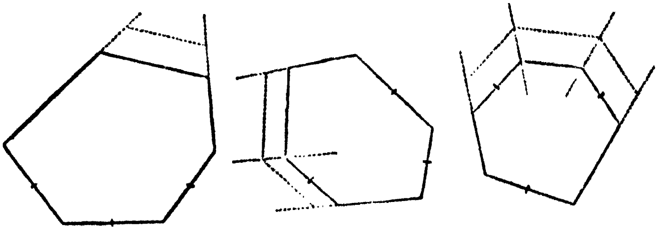


Fig. 219.

PROPOSITION XVIII.

309. Theorem.—*Two polygons of the same number of sides, having two adjacent sides and the diagonals drawn from the included angle, in the one, respectively equal to the corresponding parts in the other, and their corresponding included angles equal, are equal figures.*

Dem.—The demonstration is based upon (***284***). Let the student draw the figures, and make the applications.

PROPOSITION XIX.

310. Theorem.—*Two polygons of the same number of having all the parts (sides and angles) of the one respectively equal to the corresponding parts of the other, except two parts, are equal, unless the excepted parts are parallel sides.*

DEM.—The demonstration can be supplied by the pupil, as it is similar to the several preceding. The cases will be, 1st, When two angles are excepted, (*a*) they being consecutive, (*b*) they not being consecutive;—2d, An angle and a side, (*a*) consecutive, (*b*) not consecutive;—3d, Two sides, (*a*) consecutive, (*b*) not consecutive.

EXERCISES.

1. ***Prob.***—*Having two sides and their included angle given, to construct a triangle.*

SUG'S.—The student should draw two lines on the blackboard, and a detached angle, as the given parts. Then, making an angle equal to the given angle (***200***), he should lay off the given sides from the vertex on the sides of the angle, and join their extremities. The triangle thus formed is the one required, for any other triangle formed with these two sides and this angle will be just like this by (***284***).

2. ***Prob.***—*Having two angles and their included side given, to construct a triangle.*

3. ***Prob.***—*Having the three sides of a triangle given, to construct the triangle.*

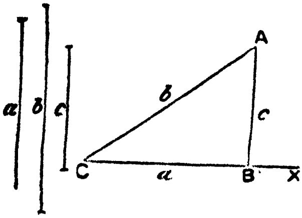
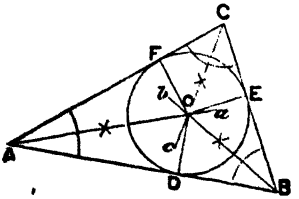


FIG. 220.

SOLUTION.—Let a, b, and c, be the given sides. Draw an indefinite line CX, and on it take CB $= a$. From C as a centre with b as a radius, describe an arc as near as can be discerned where the angle A will fall. From B, with a radius c, describe an arc intersecting the former. Then is ABC the triangle required, since any other triangle having the same sides would be equal to ABC (***292***).

4. ***Prob.***—*To inscribe a circle in a given triangle.*

SOLUTION.—For the method of doing it see PART I. (***79***). To prove the method correct, we observe that the triangles ODB and OBE have OB common, and are mutually equiangular; hence they are equal, and OD = OE. In like manner triangle OEC = OFC, and OE = OF. [Triangle OFA = ODA; but we do not need the fact in the demonstration.] Since OD = OE = OF, the circumference struck from O as a centre with a radius OD, passes through E and F. Moreover, since each side of the triangle is perpendicular to a radius at its extremity, it is tangent to the circle (***172***); and the circle is inscribed.



5. ***Prob.—Having two sides and an angle opposite one of them given, to construct the triangle.***

Solution.—1st. When the given angle is right or obtuse, the side opposite must be greater than the side adjacent, as the greatest side is opposite the greatest angle (**224**), and the greatest angle in such a triangle is the right or obtuse angle. In this case let *m* and *o* be the given sides, and O the angle opposite *o*. Draw an indefinite line O′X, construct O′ equal to O, and take O′N′ equal to *m*. From N′ as a centre, with a radius equal to O, describe an arc cutting O′X, as at M′. Draw N′M′. Then is N′M′O′ the triangle required, since all triangles having their corresponding parts equal to *m*′, *o*′, and O′ are equal.

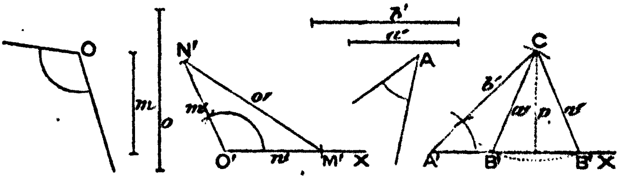


Fig. 221.

2d. When the given angle is acute, as A, there will be *no solution* if the given side, *a*, opposite A, is less than the perpendicular; *one solution* if $a = p$, or if $a >$ than both *p* and *b*, and *two solutions* if $a > p$, and less than *b*. This will appear from the construction, which is the same as in Case 1st.

6. If a perpendicular be let fall from the right angle C of the triangle ACB upon the hypotenuse, as CD, show from (**222**) that the three triangles in the figure are mutually equiangular.

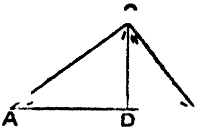


Fig. 222.

7. Given the sides of a triangle, as 15, 8, and 5, to construct the triangle.

8. Given two sides of a triangle $a = 20$, $b = 8$, and the angle B opposite the side *b* equal ⅓ of a right angle,[*] to construct the triangle.

9. Same as in the 8th, except $b = 12$. Same, except that $b = 25$.

10. Construct a triangle with angle A = ⅔ of a right angle, angle B = ⅓ of a right angle, and side *a* opposite angle A, 15.

11. Construct a right angled triangle whose hypotenuse is 16, and

---

[*] To construct this angle, bisect an angle of an equilateral triangle.

one of the other sides 7. The same with one acute angle $\frac{2}{3}$ of a right angle, and a side about the right angle 12. Will there be any difference in the *shape* of the triangles if one is constructed with the given angle adjacent to the given side, and the other with it opposite? Will there be any difference in the *size?*

12. Construct a right angled triangle having its hypotenuse 20, and one acute angle $\frac{1}{3}$ of a right angle.

13. Construct a quadrilateral three of whose sides are 20, 12, and 15, and the angle included between 20 and the unknown side $\frac{2}{3}$ of a right angle, and that between 15 and the unknown side $\frac{1}{2}$ a right angle.

SUG'S.—Make $\mathsf{A} = \frac{2}{3}$ of a right angle, and $b = 20$. From **D** as a centre, with a radius 12, strike the arc *on*. At any point on side a, make an angle $\mathsf{B}' = \frac{1}{2}$ a right angle. Take $\mathsf{B}'m = 15$, and draw **C***m* parallel to **AB′**. From the intersection **C** draw **CB** parallel to *m***B′**. Draw **CD**. Then is **ABCD** the quadrilateral required.

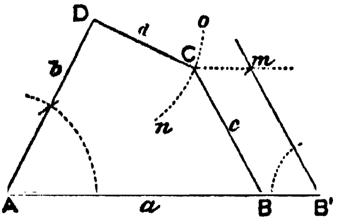


FIG. 228.

Queries.—If $d + c$ is less than the perpendicular from **D** upon **AB**, then what? If equal to the perpendicular, then what? Is it necessary to consider angle **B** in answering the two preceding queries?

14. Construct a parallelogram whose two adjacent sides are 6 and 8, and whose included angle equals $1\frac{1}{3}$ right angles.

15. Construct a heptagon whose sides in order are $a = 4$, $b = 5$, $c = 5$, $d = 6$, $e = 6$, $f = 3$, $g = 4$; and the angle included between a and b, $1\frac{1}{3}$ right angles; between b and c, $1\frac{2}{3}$; c and d, $1\frac{1}{3}$; d and

SUG'S.—See *Fig.* 187. Proceed in order, laying off the parts as given, from **A** to **F**. Draw **AF**. From **F** as a centre, with a radius $f = 3$, strike an arc, and also from **A**, with a radius $g = 4$. The intersection of these arcs will determine **G**.

Queries.—What is the limit of the sum of the possible values of the given angles? What the limit of the sum of the sides included between the unknown

SYNOPSIS.

- OF EQUALITY.
  - What? How shown?
  - PROP. I. Of straight lines.
  - PROP. II. Of circles.
  - ANGLES.
    - PROP. III. Sides parallel. Direction same or opposite.
    - PROP. IV. " " " one same, other opposite.
    - PROP. V. " perpendicular.
  - TRIANGLES.
    - PROP. VI. Two sides and included angle. { *Sch.* All parts equal.
    - PROP. VII. Two angles and included side.
      - *Cor.* Side, one adjacent and one opposite angle equal.
      - *Sch.* Exception.
    - PROP. VIII. Two sides and angle opposite one.
      - *Sch.* 1. When isosceles.
      - *Sch.* 2. When ambiguous.
    - PROP. IX. Three sides.
      - *Cor.* Equal angles opposite equal sides.
      - *Sch.* Case of obtuse angle. Form of *Fig.*
    - PROP. X. Two sides equal, included angles unequal. { *Cor.* Converse.
    - RIGHT ANGLED.
      - PROP. XI. Hypotenuse and one side.
      - PROP. XII. Hypotenuse and one acute angle.
      - PROP. XIII. Side and one acute angle.
  - QUADRILATERALS.
    - PROP. XIV. Three sides and non-included angles equal.
    - PROP. XV. Two parallelograms having two sides and the included angles equal. { *Cor.* Rectangles of same base and altitude.
  - POLYGONS OF MORE THAN 4 SIDES.
    - PROP. XVI. Three angles excepted. { *Cor.* Quadrilaterals.
    - PROP. XVII. Two angles and one side excepted.
      - *Sch.* 1. The ambiguous case.
      - *Cor.* Quadrilaterals.
      - *Sch.* 2. Three sides excepted.
    - PROP. XVIII. Two sides and included diagonals.
    - PROP. XIX. Any two parts excepted.
  - EXERCISES.
    - *Prob.* In a triangle, given two sides and included angle.
    - *Prob.* " " " angles " side.
    - *Prob.* " " " sides and angle opposite one.
    - *Prob.* " " " three sides.
    - *Prob.* To inscribe a circle in a triangle.

SECTION IX.

OF EQUIVALENCY AND AREA.

311. Equivalent Figures are such as are equal in magnitude.

PROPOSITION I.

312. Theorem.*—Parallelograms having equal bases and equal altitudes are equivalent.*

Dem.—Let ABCD and EFGH be two parallelograms having equal bases, BC and FG, and equal altitudes; then are they equivalent.

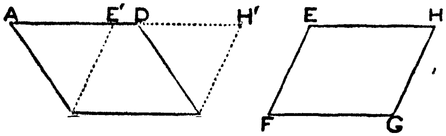


FIG. 223.*

BC; and, since the altitudes are equal, the upper base EH will fall in AD or AD produced, as E'H'. Now, the two triangles AE'B and DH'C are equal, because the three sides of the one are respectively equal to the three sides of the other. Thus AB = DC, being opposite sides of the same parallelogram. For a like reason, E'B = H'C. Also, E'H' = BC = AD. From AH' taking E'H', AE' remains, and taking AD, DH' remains. Therefore AE' = DH'. These triangles being equal, the quadrilateral ABCH' − the triangle AE'B = ABCH' − DH'C. But ABCH' − AE'B = E'BCH' = EFGH; and ABCH' − DH'C = ABCD. Hence, ABCD = EFGH. Q. E. D.

313. Cor.—*Any parallelogram is equivalent to a rectangle having the same base and altitude.*

PROPOSITION II.

Theorem.*—A triangle is equivalent to one-half of any parallelogram having an equal base and an equal altitude with the triangle.*

Dem.—Let **ABC** be a triangle. Through **C** draw **CD** parallel to **AB**; and through **A** draw **AD** parallel to **BC**. Then is **ABCD** a parallelogram, of which **ABC** is one-half (***243***). Now, as any other parallelogram having an equal base and altitude with **ABCD** is equivalent to **ABCD** (***312***), **ABC** is equivalent to one-half of *any* parallelogram having an equal base and altitude with **ABC**. Q. E. D.

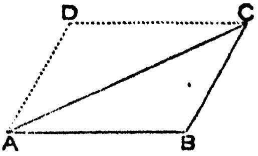


FIG. 224.

315. Cor. 1.—*A triangle is equivalent to one-half of a rectangle having an equal base and an equal altitude with the triangle.*

316. Cor. 2.—*Triangles of equal bases and equal altitudes are equivalent*, for they are halves of equivalent parallelograms.

PROPOSITION III.

317. ***Theorem.***—*The square described on a line is equivalent to four times the square described on half the line, nine times the square described on one-third the line, sixteen times the square on one-fourth the line*, etc.

Dem.—Let **AB** be any line. Upon it describe the square **ABCD**. Bisect **AB**, as at *d*, and **AD**, as at *a*. Draw *dc* parallel to **AD**, and *ab* parallel to **AB**. Now, the four quadrilaterals thus formed are parallelograms by construction, hence their opposite sides and angles are equal; and as **A**, **B**, **C**, and **D** are right angles, and **A***a* = **A***d* = *d***B** = *b***B** = etc., the four figures 1, 2, 3, 4, are equal squares. Hence **A***doa* = $\frac{1}{4}$ **ABCD**. In like manner it can be shown that the nine figures into which the square on **A'B'** is divided by drawing through the points of trisection of the sides, lines parallel to the other sides, are equal squares. Hence **A'***o'*, the square on $\frac{1}{3}$ of **A'B'**, is $\frac{1}{9}$ of the square **A'B'C'D'**. The same process of reasoning can be extended at pleasure, showing that the square on $\frac{1}{4}$ a line is $\frac{1}{16}$ the square of the whole, etc.

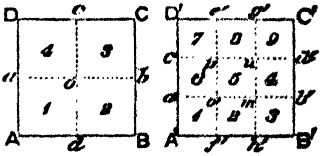


FIG. 225.

PROPOSITION IV.

318. ***Theorem.***—*A trapezoid is equivalent to two triangles having for their bases the upper and lower bases of the trapezoid, for their common altitude the altitude of the trapezoid.*

DEM.—By constructing any trapezoid, and drawing either diagonal, the student can show the truth of this theorem.

PROPOSITION V.

319. *Prob.—To reduce any polygon to an equivalent triangle.*

SOLUTION.—Let ABCDEF be a polygon which it is proposed to reduce to an equivalent triangle. Produce any side, as BC, indefinitely. Draw the diagonal EC and DH parallel to it. Draw EH. Now, consider the triangle CDE as cut off from the polygon and replaced by CHE. The magnitude of the polygon will not be changed, since CDE and CHE have the same base CE, and the same altitude, as their vertices lie in DH parallel to EC. From the polygon thus reduced we cut the triangle FHE, and replace it by its equivalent FHI, by drawing the diagonal FH, and the parallel EI. In like manner, by drawing FB and the parallel AC, we can replace FBA by its equivalent FGB. Hence, CFI is equivalent to ABCDEF. It is evident that a similar process would reduce a polygon of any number of sides to an equivalent triangle.

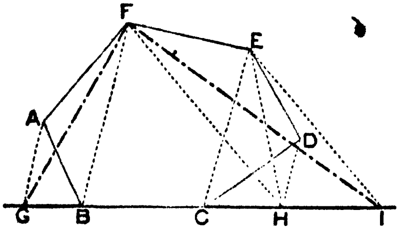


FIG. 226.

AREA.

PROPOSITION VI.

320. *Theorem.—The area of a rectangle is equal to the product of its base and altitude.*

DEM.—Let ABCD be a rectangle, then is its area equal to the base AB multiplied by the altitude AC.

If the sides AB and AC are commensurable, take some unit of length, as E, which is contained a whole number of times in each, as five times in AC, and eight times in AB, and apply it to the lines, dividing them respectively into five and eight equal parts. From the several points of division draw lines through the rectangle perpendicular to its sides. The rectangle will be divided into small parallelograms, which are all equal squares, as the angles are all right angles, and the sides all

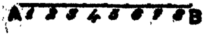


FIG. 227.

equal to each other. Each square is a unit of surface, and the area of the rectangle is expressed by the number of these squares, which is evidently equal to the number in the row on AB, multiplied by the number of such rows, or the number of linear units in AB multiplied by the number in AD.

If the two sides of the rectangle are not commensurable, take some very small unit of length which will divide one of the sides, as AC, and divide the rectangle into squares as before; the number of these squares will be the measure of the rectangle, except a small part along one side, not covered by the squares. By taking a still smaller unit, the part left unmeasured by the squares will be still less, and by diminishing the unit of length E, we can make the part unmeasured as small as we choose. It may, therefore, be made infinitely small by regarding the unit of measure as infinitesimal, and consequently is to be neglected.* Hence, in any case, the area of a rectangle is equal to the product of its base into its altitude. Q. E. D.

321. COR. 1.—*The area of a square is equal to the second power of one of its sides*, as in this case the base and altitude are equal.

322. COR. 2.—*The area of any parallelogram is equal to the product of its base into its altitude;* for any parallelogram is equivalent to a rectangle of the same base and altitude (***313***).

323. COR. 3.—*The area of a triangle is equal to one-half the product of its base and altitude;* for a triangle is one-half of a parallelogram of the same base and altitude (***314***).

324. COR. 4.—*Parallelograms or triangles*† *of equal bases are to each other as their altitudes; of equal altitudes, as their bases; and in general they are to each other as the products of their bases by their altitudes.*

PROPOSITION VII.

325. ***Theorem.***—*The area of a trapezoid is equal to the product of its altitude into one-half the sum of its parallel sides, or, what is the same thing, the product of its altitude and a line joining the middle points of its inclined sides.*

---

* This principle may be thus stated: An infinitesimal is a quantity conceived, *and to be treated*, as less than any assignable quantity; hence, as added to or subtracted from finite m, it has no value. Thus, suppose $\frac{m}{n} = a$, m, n, and a being finite quantities. Let c an infinitesimal; then $\frac{m \pm c}{n}$, or $\frac{m}{n \pm c}$, or $\frac{m \pm c}{n \pm c}$, is to be considered as still equal to a, for to consider it to differ from a by any amount we might name, would be to assign value to c.

† By this is meant the *areas* of the figures.

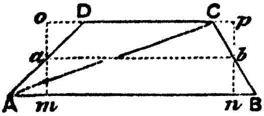


FIG. 228.

DEM.—In the trapezoid ABCD draw either diagonal, as AC. It is thus divided into two triangles, whose areas are together equal to one-half the product of their common altitude (the altitude of the trapezoid), into their bases DC and AB, or this altitude into $\frac{1}{2}(\text{AB} + \text{DC})$.

Secondly, if ab be drawn bisecting AD and CB, then is $ab = \frac{1}{2}(\text{AB} + \text{CD})$. For, through a and b draw the perpendiculars om and pn, meeting DC produced when necessary. Now, the triangles $ao\text{D}$ and $\text{A}am$ are equal, since $\text{A}a = a\text{D}$, angle $o = m$, both being right, and angle $oa\text{D} = \text{A}am$ being opposite. Whence $\text{A}m = o\text{D}$. In like manner we may show that $\text{C}p = n\text{B}$. Hence, $ab = \frac{1}{2}(op + mn) = \frac{1}{2}(\text{AB} + \text{DC})$; and area ABCD, which equals *altitude* into $\frac{1}{2}(\text{AB} + \text{DC})$, = *altitude* into ab. Q. E. D.

PROPOSITION VIII.

326. Theorem.—*The area of a regular polygon is equal to one-half the product of its apothem into its perimeter.*

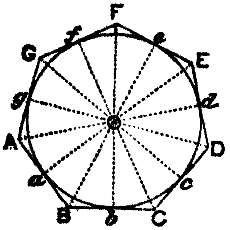


FIG. 229.

DEM.—Let ABCDEFG be a regular polygon whose apothem is $\text{O}a$; then is its area equal to $\frac{1}{2}\,\text{O}a\,(\text{AB} + \text{BC} + \text{CD} + \text{DE} + \text{EF} + \text{FG} + \text{GA})$.

Drawing the inscribed circle, the radii $\text{O}a$, $\text{O}b$, etc., to the points of tangency, and the radii of the circumscribed circle OA, OB, etc. (**264**, **265**), the polygon is divided into as many equal triangles as it has sides. Now, the apothem (or radius of the inscribed circle) is the common altitude of these triangles, and their bases make up the perimeter of the polygon. Hence, the area $= \frac{1}{2}\,\text{O}a\,(\text{AB} + \text{BC} + \text{CD} + \text{DE} + \text{EF} + \text{FG} + \text{GA})$. Q. E. D.

327. COR.—*The area of any polygon in which a circle can be inscribed is equal to one-half the product of the radius of the inscribed circle into the perimeter.*

The student should draw a figure and observe the fact. It is especially worthy of note in the case of a triangle. See *Fig.* 60.

PROPOSITION IX.

Theorem.—*The area of a circle is equal to one-half the product of its radius into its circumference.*

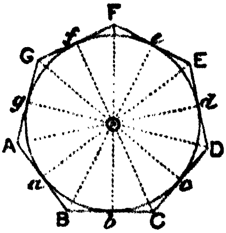


FIG. 230.

DEM.—Let O*a* be the radius of a circle. Circumscribe any regular polygon. Now the area of this polygon is one-half the product of its apothem and perimeter. Conceive the number of sides of the polygon, indefinitely increased, the polygon still continuing to be circumscribed. The apothem continues to be the radius of the circle, and the perimeter approaches the circumference. When, therefore, the number of sides of the polygon becomes infinite, it is to be considered as coinciding with the circle, and its perimeter with the circumference. Hence the area of the circle is equal to one-half the product of its radius into its circumference. Q. E. D.

329. DEF.—***A Sector*** is a part of a circle included between two radii and their intercepted arc. *Similar Sectors* are sectors in different circles, which have equal angles at the centre.

330. COR. 1.—*The area of a sector is equal to one-half the product of the radius into the arc of the sector.*

331. COR. 2.—*The area of a sector is to the area of the circle as the arc of the sector is to the circumference, or as the angle of the sector is to four right angles.*

EXERCISES.

1. What is the area in acres of a triangle whose base is 75 rods and altitude 110 rods?

2. What is the area of a right angled triangle whose sides about the right angle are 126 feet and 72 feet?

3. If 2 lines be drawn from the vertex of a triangle to the dividing the base into parts which are to each other as 2, 3, and 5, how is the triangle divided? How does a line drawn from an angle to the middle of the opposite side divide a triangle?

4. Review the exercises on pages 49 and 50, giving the reasons, in each case.

SYNOPSIS.

- EQUIVALENCY AND AREA.
  - EQUIVALENCY.
    - Definition.
    - PROP. I. Of parallelograms.
      - *Cor.* Paral. and rectangle.
    - PROP. II. Of triangles.
      - *Cor.* 1. Triangle and rectangle.
      - *Cor.* 2. Of equal bases and equal altitudes.
    - PROP. III. Square on ½, ⅓, ¼ a line, etc.
    - PROP. IV. Trapezoid.
    - PROB. V. To reduce a polygon to a triangle.
  - AREA.
    - PROP. VI. Of rectangle.
      - *Cor.* 1. Of square.
      - *Cor.* 2. Any parallelogram.
      - *Cor.* 3. Of triangle.
      - *Cor.* 4. Relation of parallelograms and of triangles.
    - PROP. VII. Of trapezoid.
    - PROP. VIII. Of regular polygons.
      - *Cor.* Of any circumscribed polygon.
    - PROP. IX. Of a circle.
      - *Def.* Of sector.
      - *Cor.* 1. Area of sector.
      - *Cor.* 2. Relation of sector to circle.
  - EXERCISES.

SECTION X.

OF SIMILARITY.

332. The primary notion of similarity is *likeness of form*. Two figures are said to be similar which have the same shape, although they may differ in magnitude.* A more scientific definition is as follows:

333. ***Similar Figures*** are such as have their angles respectively equal, and their homologous sides proportional.

334. ***Homologous Sides*** of similar figures are those which are included between equal angles in the respective figures.

---

* The student should be careful, at the outset, to mark the fact that similarity involves two things, EQUALITY OF ANGLES and PROPORTIONALITY OF SIDES. It will appear that, in the case of triangles, if one of these facts exists, the other does also; but this is not so in other polygons, as is illustrated in PART I.

IN SIMILAR TRIANGLES, THE HOMOLOGOUS SIDES ARE THOSE OPPOSITE THE EQUAL ANGLES.

PROPOSITION I.

335. Theorem.*—Triangles which are mutually equiangular are similar.*

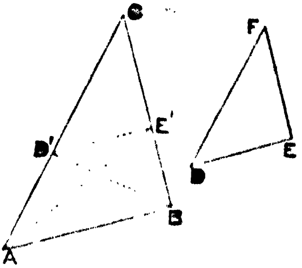


Fig. 221.

DEM.—Let ABC and DEF be two mutually equiangular triangles, in which A = D, B = E, and C = F; then are the sides opposite these equal angles proportional, and the triangles possess both requisites of similar figures; *i. e.*, they are mutually equiangular and have their homologous sides proportional, and are consequently similar.

To prove that the sides opposite the equal angles are proportional, place the triangle DEF upon ABC, so that F shall coincide with its equal C, CE′ = FE, and CD′ = FD. Draw AE′, and D′B. Since angle CE′D′ = CBA, D′E′ is parallel to AB, and as the triangles D′E′A and D′E′B have a common base D′E′ and the same altitude, their vertices lying in a line parallel to their base, they are equivalent (*316*). Now, the triangles CD′E′ and D′E′A, having a common altitude, are to each other as their bases (*324*). Hence,

CD′E′ : D′E′A : : CD′ : D′A.

For like reason CD′E′ : D′E′B : : CE′ : E′B.

Then, since D′E′A and D′E′B are equivalent, the two proportions have a common ratio, and we may write CD′ : D′A : : CE′ : E′B.

By composition CD′ : CD′ + D′A : : CE′ : CE′ + E′B,
or CD′ : CA : : CE′ : CB, or FD : CA : : FE : CB.

In a similar manner, by applying angle E to B, we can show that FE : CB : : ED : BA. Therefore, FD : CA : : FE : CB : : ED : BA. Q. E. D.

336. COR. 1.—*If two triangles have two angles of the one [illegible]tively equal to two angles of the other, the third angles being equal [illegible]), the triangles are similar.*

337. COR. [illegible].—*A line drawn through a triangle parallel to any side divides the other sides proportionally.*

Thus D′E′ being parallel to AB, it is shown in the proposition that CD′ : D′A : : CE′ : E′B.

338. Cor. 3.—*If any two lines cut a series of parallels, they are divided proportionally.*

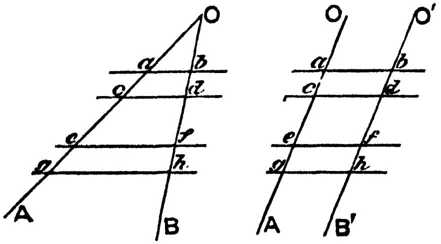


Fig. 232.

Dem.—If the two secant lines are parallel, as OA and O'B', the intercepted parts are equal, *i. e.*, $ac = bd$, $ce = df$, $eg = fh$, etc. (**242**). Hence, $ac : bd :: ce : df :: eg : fh$. Secondly, if the secant lines are not parallel, let them meet in some point, as O. Then, by the proposition, we have

$$Oa : ac :: Ob : bd \quad (1), \qquad \text{and also } Oc : ce :: Od : df \quad (2).$$

Taking the first by composition, it becomes

$$Oa + ac : ac :: Ob + bd : bd, \text{ or } Oc : ac :: Od : bd \quad (3).$$

Now, as the antecedents in (2) and (3) are the same, we have

$$ac : bd :: ce : df, \text{ or } ac : ce :: bd : df.$$

In like manner, we may show that

$$ce : df :: eg : fh, \text{ or } ce : eg :: df : fh.$$

the triangles **ABC** and **DEF** are mutually equiangular, and consequently similar. Q. E. D.

340. SCH.—As we now know that if two triangles are mutually equiangular, they are similar; or, if they have their sides proportional, they are similar, it will be sufficient hereafter, in any given case, to prove *either one* of these facts, in order to establish the similarity of two triangles. For, either fact being proved, the other follows as a consequence. See Section VI., PART I., for familiar illustrations of this most important subject.

PROPOSITION III.

341. Theorem.*—Two triangles which have the sides of the one respectively parallel or perpendicular to the sides of the other, are similar.*

DEM.—Let **ABC** and **A'B'C'** be two triangles whose sides are respectively parallel or perpendicular to each other, then are the triangles similar.

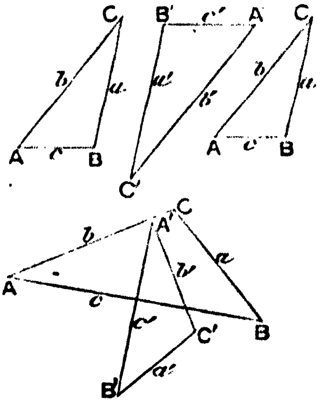


FIG. 234.

For, any angle in one triangle is either equal or supplementary to the angle in the other which is included between the sides which are parallel or perpendicular to its own sides. Thus **A** either equals **A'**, or **A** + **A'** = 2 right angles (***281, 282, 283***). Now, if the corresponding angles are all supplementary, that is, if **A** + **A'** = 2 R.A., **B** + **B'** = 2 R.A., and **C** + **C'** = 2 R.A., the sum of the angles of the two triangles is 6 right angles, which is impossible. Again, if one angle in one triangle equals the corresponding angle in the other, as **A** = **A'**, and the other angles are supplementary, the sum is 4 right angles plus twice the equal angle, which is impossible. Hence, two of the angles of one triangle must be equal respectively to two angles of the other; and, if two are equal, the third angle in one is equal to the third in the other (***221***). Hence, the triangles are mutually equiangular, and therefore similar (***335***). Q. E. D.

PROPOSITION IV.

342. Theorem.*—Two triangles, which have an angle in equal, and the sides about the equal angles proportional, are similar*

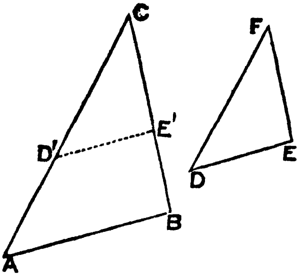


FIG. 235.

DEM.—In the triangles ABC and DEF let C = F, and AC : DF :: CB : FE; then are the triangles similar.

For, place F on its equal C, and let D fall at D'. Draw D'E' parallel to AB. Then AC : D'C (= DF) :: BC : CE' (337). But by hypothesis AC : DF :: BC : FE. ∴ CE' = FE, and the triangles D'CE' and DFE are equal (284). Therefore, D'CE' being equiangular with ACB, is similar to it (335); and as DFE is equal to D'CE', DFE is similar to ACB. Q. E. D.

PROPOSITION V.

343. Theorem.*—In any right angled triangle, if a line be drawn from the right angle perpendicular to the hypotenuse, it divides the triangle into two triangles, which are similar to the given triangle, and consequently similar to each other.*

DEM.—Let ACB be a triangle right-angled at C, and CD a perpendicular upon the hypotenuse AB; then are ACD and CDB similar to ACB, and consequently to each other.

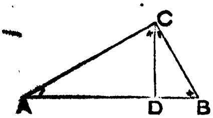


FIG. 236.

For, the triangles ACD and ACB have the angle A common, and a right angle in each; hence they are mutually equiangular, and consequently similar(*336*). For a like reason CDB and ACB are similar. Finally, as ACD and CDB are both similar to ACB, they are similar to each other. Q. E. D.

344. COR. 1.—*Either side about the right angle is a mean proportional between the whole hypotenuse and the adjacent segment.*

DEM.—This is a direct consequence of the similarity of the partial triangles with the whole triangle. Thus, comparing the homologous sides of ACD and ACB, we have AD : AC :: AC : AB;* and from CDB and ACB, we have DB : CB :: CB : AB.

345. COR. 2.—*The perpendicular is a mean proportional between the segments of the hypotenuse.*

DEM.—This is a consequence of the similarity of ACD and CDB. Thus, AD : CD :: CD : DB.

---

* Notice that AD of the triangle ACD is opposite angle ACD, and AC, its consequent, is of the triangle ACB, and opposite the angle B, which equals angle ACD. The student must be sure that he knows in what order to take the sides, and why.

Queries.—To which triangle does the first CD belong? To which the second? Why is CD made the consequent of AD? Why, in the second ratio, are CD and DB to be compared?

346. Cor. 3.—*The square described on the hypotenuse of a right angled triangle is equivalent to the sum of the squares described on the other two sides.*

Dem.—From *Cor.* 1, $\overline{AC}^2 = AB \times AD$

and also $\overline{CB}^2 = AB \times DB$.

Therefore, adding, $\overline{AC}^2 + \overline{CB}^2 = AB\,(AD + DB) = \overline{AB}^2$.

347. Cor. 4.—*If a perpendicular be let fall from any point in a circumference upon a diameter, this perpendicular is a mean proportional between the segments of the diameter.*

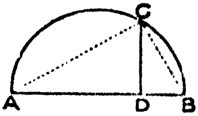


Fig. 237.

Dem.—Thus, AD : CD :: CD : DB, or $\overline{CD}^2 = AD \times DB$.

For, drawing AC and CB, ACB is a right angle, and the case falls under *Cor.* 2.

The chords AC and CB are mean proportionals between the whole diameter and their adjacent segments by *Cor.* 1.

348. Sch.—This proposition, with its corollaries, is perhaps the most fruitful in direct practical results of any in Geometry. *Cor.* 3 will be recognized as a demonstration of the Pythagorean proposition (***109***), Part I. There are many other demonstrations of exceeding beauty, some of which will be given in Part III. The one here given is the simplest, and shows best the way in which this truth grows out of the more general fact of similarity.

PROPOSITION VI.

***349.* Theorem.**—*Regular polygons of the same number of sides are similar figures.*

Dem.—Let P and P′ be two regular polygons of the same number of sides,* a, b, c, d, etc., being the sides of the former, and a', b', c', d', etc., the sides of the latter. Now, by the definition of regular polygons, the sides a, b, c, d, etc., are equal each to each, and also a', b', c', d', etc. Hence, we have $a : a' :: b : b' :: c : c' :: d : d'$, etc. Again, the angles are equal, since n being the number of sides of each polygon, each angle is

$$\frac{n \times 2 \text{ right angles} - 4 \text{ right angles}}{n} \quad (\mathbf{256}).$$

Hence the polygons are mutually equiangular, and have their sides proportional; that is, they are similar. Q. E. D.

---

* The student may construct two regular hexagons, if thought desirable.

350. COR. 1.—*The corresponding diagonals of regular polygons of the same number of sides are in the same ratio as the sides of the polygons.*

Let the student draw a figure and demonstrate the fact.

351. COR. 2.—*The radii of the inscribed, and also of the circumscribed circles, of regular polygons of the same number of sides, are in the same ratio as the sides of the polygons.*

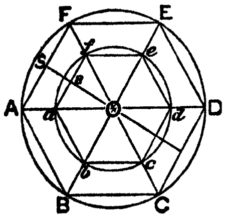


FIG. 238.

DEM.—Since the angles F and *f* are equal, and bisected by FO, the right angled triangles OSF, O*sf* are equiangular, and hence similar. Therefore FS : *fs* : : SO : *s*O or FO : *f*O. Whence, doubling both terms of the first couplet,

FA : *fa* : : SO : *s*O or FO : *f*O.

PROPOSITION VII.

352. ***Theorem.***—*Circles are similar figures.*

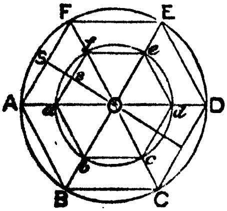


FIG. 239.

DEM.—Let O*a* and OA be the radii of any two circles. Place the circles so that they shall be concentric, as in the figure. Inscribe the regular hexagons, as *abcdef*, ABCDEF. Conceive the arcs AB, BC, etc., of the outer circumference, bisected, and the regular dodecagon inscribed, and also the corresponding regular dodecagon in the inner circumference. These are similar figures by (***349***). Now, as the process of bisecting the arcs of the exterior circumference can be conceived as indefinitely repeated, and the corresponding *regular* polygons as inscribed in each circle, the circles may be considered as regular polygons of the same number of sides, and hence similar. Q. E. D.

353. COR.—Arcs of similar sectors are to each other as the radii of their circles; *i. e.*, arc *fe* : arc FE : : O*f* : OF.

SCH.—The circle is said to be the *Limit* of the inscribed polygon, and the circumference the *limit* of the perimeter. By this is meant that as the number of the sides of the inscribed polygon is increased it approaches nearer and nearer to equality with the circle. The apothem approaches equality with the radius, and hence has the radius for its limit. It is an axiom of great importance in mathematics that, *Whatever can be shown to be true of*
tude as it approaches its limit indefinitely, is true of that limit.

EXERCISES.

1. ***Prob.***—*To divide a given line into parts which shall be proportional to several given lines.*

Solution.—Let it be required to divide **OP** into parts proportional to the lines **A**, **B**, **C**, and **D**. Draw **ON** making any convenient angle with **OP**, and on it lay off **A**, **B**, **C**, and **D**, in succession, terminating at **M**. Join **M** with the extremity **P**, and draw parallels to **MP** through the other points of division. Then by reason of the parallels we shall have

A : **B** : **C** : **D** :: $a : b : c : d$, (*338*).

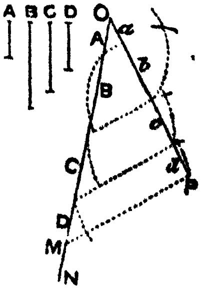


Fig. 240.

2. ***Prob.***—*To find a fourth proportional to three given*

For the solution see (*89*). Repeat the process, and give the reasons.

3. ***Prob.***—*To find a third proportional to two given lines.*

Solution.—This may be solved as the two preceding. Thus, take any two lines, as **A** and **B**, for the given lines. We are to find a third line x, such that **A** : **B** :: **B** : x. The figure will suggest the details.

The following is a solution based on (*347*). Draw an indefinite line **AM**. Take **AD** = **A**, and erect **BD** = **B**. Join **AB**, and bisect it by the perpendicular **ON**. Then with **O** as a centre, and **OA** as a radius, describe a semi-circumference. This will pass through **B**. (Why?) Also **AD** : **BD** :: **BD** : **CD** (= x). (Why?)

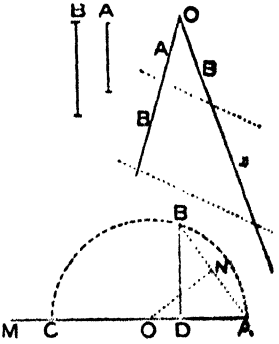


Fig. 241.

4. Draw any straight line on the blackboard, and divide it into equal parts, upon the principle used in the preceding solutions.

5. Review the exercises under (*89, 90*), and give the reasons.

6. ***Prob.***—*To find a mean proportional between two given lines.*

For the solution see (*110*). Repeat the process, and give the reasons for the method.

7. **DE** being parallel to **BC**, prove that the triangles **DOE** and **BOC** are similar, and hence that **OD** : **OC** :: **OE** : **OB**. Are the following proportions true?

OD : **OC** :: **OE** : **OB**, **OD** : **DE** :: **OC** : **BC**,
OD : **OE** :: **OC** : **OB**, **OB** : **BC** :: **OE** : **DE**.

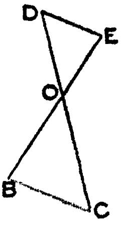


Fig. 242.

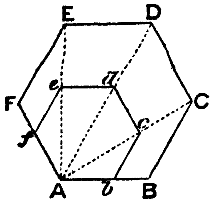


Fig. 243.

8. Show that if ABCDEF is a regular polygon, *Abcdef* is also regular *bc*, *cd*, etc., being parallel to BC, CD, etc Show that any two similar polygons may be *placed in similar relative positions*, and hence show that the corresponding diagonals are in the same ratio as the homologous sides.

9. The sides of one triangle are 7, 9, and 11. The side of a second similar triangle, homologous with side 9, is 4½. What are the other sides of the latter?

10. The diameter of a circle is 20. What is the perpendicular distance to the circumference from a point in the diameter 15 from one extremity? What are the distances from the point where this perpendicular meets the circumference to the extremities of the diameter?

SYNOPSIS.

Of Similarity.

- Primary notion of similarity.
- Definition of similarity.
- Homogeneity of sides. In general. In triangles.
- **Conditions of Similarity in Triangles.**
  - Prop. I. Mutually equiangular.
    - *Cor.* 1. Two angles equal.
    - *Cor.* 2. A parallel to a side.
    - *Cor.* 3. Lines cutting parallels.
  - Prop. II. Sides proportional.
    - *Sch.* Either of two facts sufficient.
  - Prop. III. Sides parallel or perpendicular.
  - Prop. IV. An angle equal in each, and sides proportional.
  - Prop. V. Perpendicular from right angle upon hypotenuse.
    - *Cor.* 1. Side about right angle.
    - *Cor.* 2. Perpendicular.
    - *Cor.* 3. Square on hypotenuse.
    - *Cor.* 4. Perpendicular on diameter.
    - *Sch.* Importance of this *Prop.* and *Cor's.*
- Prop. VI. Regular polygons similar.
  - *Cor.* 1. Corresponding diagonals.
  - *Cor.* 2. Radii of inscribed and circumscribed circles.
- Prop. VII. Circles similar.
  - *Sch.* Circle limit of polygon.
- Exercises.
  - *Prob.* To divide a line into proportional parts.
  - *Prob.* To find a fourth proportional.
  - *Prob.* To find a third proportional.
  - *Prob.* To find a mean proportional.

SECTION XI.

APPLICATIONS OF THE DOCTRINE OF SIMILARITY TO THE DEVELOPMENT OF GEOMETRICAL PROPERTIES OF FIGURES.

354. The doctrine of similarity, as presented in the preceding section, is the chief reliance for the development of the geometrical properties of figures. This section will be devoted to the investigation of a few of the more elementary properties of plane figures, which we are able to discover by means of this doctrine.

OF THE RELATIONS OF THE SEGMENTS OF TWO LINES INTERSECTING EACH OTHER, AND INTERSECTED BY A CIRCUMFERENCE.

PROPOSITION I.

-If two chords intersect each other in a
their segments are reciprocally proportional; whence the product of the segments of one chord equals the product of the segments of the other.

DEM.—Let the chords AC and BD intersect at O; then is AO : BO :: DO : CO, whence AO × CO = BO × DO.

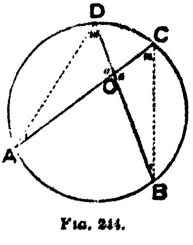


FIG. 244.

For, draw AD and BC. The two triangles AOD and BOC are equiangular, and hence similar; since the angles at O are vertical, and consequently equal (*134*), and D = C, because both are measured by ½ arc AB (*210*). (A = B because both are measured by ½ arc DC; but it is only necessary to show that *two* angles are equal in order to show that the triangles are equiangular, and hence similar.) Now, comparing the homologous sides (those opposite the equal angles), we have AO : BO :: DO : CO; whence, AO × CO = BO × DO. Q. E. D.

QUERIES.—Why is AO compared with BO? Why DO with CO? Would AO : CO :: BO : DO be true? Would AO : DO :: BO : CO? What is the force of the word "reciprocally," as used in the proposition?

PROPOSITION II.

356. Theorem.—*If from a point without a circle, two secants be drawn terminating in the concave arc, the whole secants are reciprocally proportional to their external segments; whence the product of one secant into its external segment equals the product of the other into its external segment.*

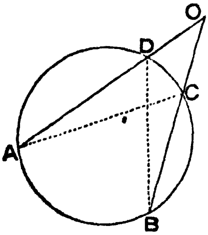


Fig. 245.

Dem.—OA and OB being secants, $OA : OB :: OC : OD$, and consequently $OA \times OD = OB \times OC$. For, drawing AC and DB, the two triangles ACO and BDO have angle O common, and $A = B$, since both are measured by $\frac{1}{2}$ DC; hence the triangles are similar, and we have $OA : OB :: OC : OD$, and consequently $OA \times OD = OB \times OC$. Q. E. D.

Same queries as under the preceding demonstration.

PROPOSITION III.

357. Theorem.—*If from a point without a circle a tangent be drawn, and a secant terminating in the concave arc, the tangent is a mean proportional between the whole secant and its external segment; whence the square of the tangent equals the product of the secant into its external segment.*

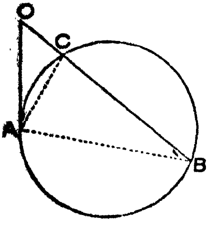


Fig. 246.

Dem.—OA being a tangent and OB a secant, $OB : OA :: OA : OC$, whence $\overline{OA}^2 = OB \times OC$. For, drawing AB and AC, the triangles OAB and ACO have angle O common, and $OAC = B$, since each is measured by $\frac{1}{2}$ arc AC; hence the triangles are similar, and $OB : OA :: OA : OC$, whence $\overline{OA}^2 = OB \times OC$. Q. E. D.

OF THE BISECTOR OF AN ANGLE OF A TRIANGLE.

PROPOSITION IV.

358. Theorem.—*A line which bisects any angle of a triangle divides the opposite side into segments proportional to the adjacent sides.*

DEM.—Let CD bisect the angle ACB; then AD : DB :: AC : CB.

For, draw BE parallel to CD, and produce it till it meets AC produced, as at E. Now, by reason of the parallels CD and EB, angle ACD = AEB, and angle DCB = CBE (*152*). Whence, as ACD = DCB by hypothesis, E = CBE, and CE = CB (*227*). Also, since CD is parallel to EB, AD : DB :: AC : CE (*337*). Substituting for CE its equal CB, we have AD : DB :: AC : CB. Q. E. D.

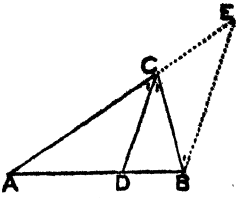


FIG. 247.

PROPOSITION V.

359. Theorem.*—If a line be drawn from any vertex of a triangle bisecting the exterior angle and intersecting the opposite side produced, the distances from the other vertices to this intersection are proportional to the adjacent sides.*

DEM.—Through the vertex C let CD be drawn, bisecting the exterior angle FCB, and intersecting AB produced in D; then AD : BD :: AC : CB.

For, draw BE parallel to CD. By reason of these parallels, angle FCB = CEB, and BCE = CBE (*152*). Hence CEB = CBE, and CB = CE. Also, by reason of the parallels, AD : BD :: AC : CE, or its equal CB (*335*). Q. E. D.

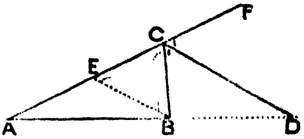


FIG. 248.

PROPOSITION VI.

360. Theorem.*—If a line be drawn bisecting any angle of a triangle and intersecting the opposite side, the rectangle of the sides about the bisected angle equals the rectangle of the segments of the third side, plus the square of the bisector.*

DEM.—Let CD bisect the angle ACB; then AC × CB = AD × DB + $\overline{CD}^2$.

For, circumscribe the circle about the triangle, produce the bisector till it meets the circumference at E, and draw EB. The triangles ADC and CBE are similar, since angle ACD = ECB, by hypothesis, and A = E, because each is measured by ½ arc CB. Therefore, AC : CE :: CD : CB, whence AC × CB = CE × CD = (DE + CD) CD = DE × CD + $\overline{CD}^2$. For DE × CD, substituting its equivalent AD × DB (*355*), we have AC × CB = AD × DB + $\overline{CD}^2$. Q. E. D.

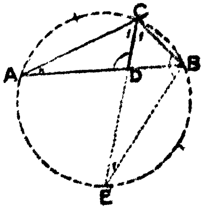


FIG. 249.

PROPOSITION VII.

361. *Theorem.*—*The bisectors of the angles of a triangle all pass through the same point, which is the centre of the inscribed circle.*

Dem.—Draw two lines bisecting two of the angles, and from their intersection draw a line to the other angle. Then show that the latter angle is bisected. By (Ex. 4, page 134) this point is shown to be the centre of the inscribed circle. [The student should fill out the demonstration.]

AREAS OF SIMILAR FIGURES.

PROPOSITION VIII.

362. *Theorem.*—*The areas of similar triangles are to each other as the squares described on their homologous sides.*

Dem.—Let ABC and DEF be any two similar triangles; then is

$$\textit{area}\ ABC : \textit{area}\ DEF :: \overline{CB}^2 : \overline{FE}^2 :: \overline{AC}^2 : \overline{DF}^2 :: \overline{AB}^2 : \overline{DE}^2$$

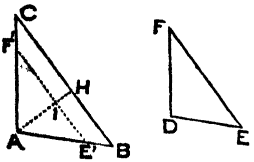


Fig. 250.

For, place the largest angle of the triangle DEF, as D, on its equal angle A, of the triangle ABC*; let E fall at E', F at F', and draw E'F'; then is triangle AE'F' = DEF (*284*), and E'F' is parallel to BC. Let fall a perpendicular from A to CB. Then AI is the altitude of AE'F', and AH of ABC. Now, by similar triangles we have $CB : F'E' :: AH : AI$.

But $\frac{1}{2}AH : \frac{1}{2}AI :: AH : AI$; and, multiplying corresponding terms, $\frac{1}{2}AH \times CB : \frac{1}{2}AI \times F'E' :: \overline{AH}^2 : \overline{AI}^2$. Whence, since $\frac{1}{2}AH \times CB =$ *area* ABC, and $\frac{1}{2}AI \times F'E' =$ *area* AE'F' = *area* DEF, and $AH : AI :: CB : FE :: AC : DF :: AB : DE$, or $\overline{AH}^2 : \overline{AI}^2 :: \overline{CB}^2 : \overline{FE}^2 :: \overline{AC}^2 : \overline{DF}^2 :: \overline{AB}^2 :: \overline{DE}^2$; we have

$$\textit{area}\ ABC : \textit{area}\ DEF :: \overline{CB}^2 : \overline{FE}^2 :: \overline{AC}^2 : \overline{DF}^2 :: \overline{AB}^2 : \overline{DE}^2. \quad \text{Q. E. D.}$$

PROPOSITION IX.

363. *Theorem.*—*The areas of similar polygons are to each other as the squares of the homologous sides of the polygons.*

---

* The only object in taking the *largest* angle is to make the perpendicular fall within the triangle. Any two equal angles may be applied, and the demonstration is essentially the same.

Dem.—Let $abcdef$ and ABCDEF be two similar polygons. Designate the former by p, and the latter by P. Then $p : \mathrm{P} :: \overline{ab}^2 : \overline{\mathrm{AB}}^2$ or as the squares of any other two homologous sides.

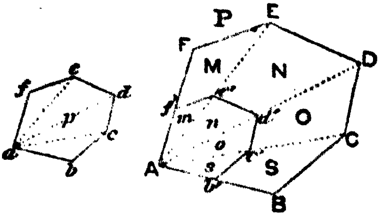


Fig. 251.

For, from the equal angles a and A drawing the diagonals, the corresponding partial angles into which a and A are divided are equal. [Let the student show why by *342.*] Now take $\mathrm{A}b' = ab$, and draw $b'c'$, making angle $\mathrm{A}b'c' = b$. Then $b'c' = bc$, and $\mathrm{A}c' = ac$, since the triangles abc and $\mathrm{A}b'c'$ have two angles and the included side of one equal to two angles and the included side of the other. In like manner draw $c'd'$ making angle $b'c'd' = bcd$, $c'd' = cd$, and $\mathrm{A}d' = ad$. So, also, making angle $c'd'e' = cde$, and angle $d'e'f' = def$, $d'e' = de$, $e'f' = ef$, and $f'\mathrm{A} = fa$. Hence the polygon $\mathrm{A}b'c'd'e'f' = p$, and its sides are respectively parallel to the corresponding sides of P. Now, let m, n, o, and s represent the triangles in which they stand, and M, N, O, and S the corresponding triangles of P, as AFE, etc. Triangles m and M being similar, and also n and N, we have

$$m : \mathrm{M} :: \overline{\mathrm{A}e'}^2 : \overline{\mathrm{AE}}^2, \text{ and } n : \mathrm{N} :: \overline{\mathrm{A}e'}^2 : \overline{\mathrm{AE}}^2.$$

Whence $m : \mathrm{M} :: n : \mathrm{N}$.

In like manner we can show that $n : \mathrm{N} :: o : \mathrm{O}$, and that $o : \mathrm{O} :: s : \mathrm{S}$.

Whence $m : \mathrm{M} :: n : \mathrm{N} :: o : \mathrm{O} :: s : \mathrm{S}$.

By composition, $(m + n + o + s)$ (or p) : $(\mathrm{M} + \mathrm{N} + \mathrm{O} + \mathrm{S})$ (or P) $:: s : \mathrm{S}$.

But $s : \mathrm{S} :: \overline{\mathrm{A}b'}^2$ (or $\overline{ab}^2$) : $\overline{\mathrm{AB}}^2$. Therefore $p : \mathrm{P} :: \overline{ab}^2 : \overline{\mathrm{AB}}^2$, or as the squares of any two homologous sides. Q. E. D.

364. Cor. 1.—*Similar polygons* are to each other as the squares of their corresponding diagonals.*

In the demonstration we have $s : \mathrm{S} :: \overline{\mathrm{A}c'}^2$ (or $\overline{ac}^2$) : $\overline{\mathrm{AC}}^2$. Whence $p : \mathrm{P} :: \overline{ac}^2 : \overline{\mathrm{AC}}^2$. The same may be shown of any other corresponding diagonals.

365. Cor. 2.—*Regular polygons* of the same number of sides are to each other as the squares of their homologous sides; since they are similar figures (**349**).*

366. Cor. 3.—*Regular polygons* of the same number of sides are to each other as the squares of their apothems.*

For their apothems are to each other as their sides. Hence the squares of their apothems are to each other as the squares of their sides.

367. Cor. 4.—*Circles are to each other as the squares of their radii (**352**), and as the squares of their diameters.*

---

* This is a common elliptical form for "The *areas* of, etc."

OF PERIMETERS AND THE RECTIFICATION OF THE CIRCUMFERENCE.

368. The Rectification of a curve is the process of finding its length.

The term *rectification* signifies making straight, and is applied as above, under the conception that the process consists in finding a straight line equal in length to the curve.

PROPOSITION X.

369. Theorem.*—The perimeters of similar polygons are to each other as their homologous sides, and as their corresponding diagonals.*

DEM.—Let a, b, c, d, etc., and A, B, C, D, etc., be the homologous sides of two similar polygons whose perimeters are p and P; then $p : \mathrm{P} :: a : \mathrm{A} :: b : \mathrm{B} :: c : \mathrm{C}$, etc.; and r and R being corresponding diagonals, $p : \mathrm{P} :: r : R$. Since the polygons are similar, $a : \mathrm{A} :: b : \mathrm{B} :: c : \mathrm{C} :: d : \mathrm{D}$, etc. By composition, $(a + b + c + d + \text{etc.})$ (or p) : $(\mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D} + \text{etc.})$ (or P) $:: a : \mathrm{A}$, or as any other homologous sides. Also, as the homologous sides are to each other as the corresponding diagonals (*350*), $p : \mathrm{P} :: r : R$. Q. E. D.

). COR. 1.—*The perimeters of regular polygons of the same number of sides are to each other as the apothems of the polygons.*

For the apothems are to each other as the sides of the polygons (*351*).

371. COR. 2.—*The circumferences of circles are to each other as their radii, and as their diameters;* since they may be considered as regular polygons of the same number of sides (*352*).

PROPOSITION XI.

372. Theorem.*—The circumference of a circle whose radius is , is 2π, the numerical value of π being approximately* 3.1416.

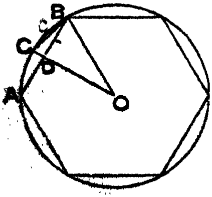


FIG. [illegible].

DEM.—We will approximate the circumference of a circle whose radius is 1, by obtaining, 1st, the perimeter of the regular inscribed hexagon; 2d, the perimeter of the regular inscribed dodecagon; 3d, the perimeter of the regular inscribed polygon of 24 sides; then of 48, etc.

In order to do this, let us find the relation between the chord of an arc and the chord of ½ the arc in a circle whose radius is [illegible] Call the chord of an arc as

AB, *C*, and the chord of half the arc, as CB, *c*. Now, BDO is right angled at D, whence $\overline{BO}^2 = \overline{BD}^2 + \overline{DO}^2$ (*346*), or $DO = \sqrt{\overline{BO}^2 - \overline{BD}^2}$. But in the present case BO = 1; hence $DO = \sqrt{1 - \frac{1}{4}C^2}$. Taking DO from CO, we have $CD = 1 - \sqrt{1 - \frac{1}{4}C^2}$ From the right angled triangle BDC we have CB (or *c*) = $\sqrt{\overline{BD}^2 + \overline{CD}^2}$, or substituting $\frac{1}{2}C$ for BD, and $1 - \sqrt{1 - \frac{1}{4}C^2}$ for CD, this reduces to $c = \sqrt{2 - \sqrt{4 - C^2}}$, or, $\left[2 - (4 - C^2)^{\frac{1}{2}}\right]^{\frac{1}{2}}$

By the use of this formula, we make the following computations:

No sides.	Form of Computation.	Length of Side.	Perimeter.
6.	See (*271*)	1.00000000	6.00000000
12.	$c = \sqrt{2 - \sqrt{4 - 1^2}} = \sqrt{2 - \sqrt{3}}$, or $(2 - 3^{\frac{1}{2}})^{\frac{1}{2}}$	.51763809	6.21165708
24.	$c = \{2 - [4 - (2 - 3^{\frac{1}{2}})]^{\frac{1}{2}}\}^{\frac{1}{2}} = [2 - (2 + 3^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}$	.26105238	6.26525722
48.	$c = (2 - \{4 - [2 - (2 + 3^{\frac{1}{2}})^{\frac{1}{2}}]\}^{\frac{1}{2}})^{\frac{1}{2}}$ $= \{2 - [2 + (2 + 3^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}\}^{\frac{1}{2}}$	.13080626	6.27870041
96.	$c = (2 - \{2 + [2 + (2 + 3^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}\}^{\frac{1}{2}})^{\frac{1}{2}}$	.06543817	6.28206396
192.	$c = [2 - (2 + \{2 + [2 + (2 + 3^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}\}^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}$	.03272346	6.28290510
384.	$c = \{2 - [2 + (2 + \{2 + [2 + (2 + 3^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}\}^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}\}^{\frac{1}{2}}$	.01636228	6.28311544
768.	$c = (2 - \{2 + [2 + (2 + \{2 + [2 + (2 + 3^{\frac{1}{2}})]^{\frac{1}{2}}\}^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}\}^{\frac{1}{2}})^{\frac{1}{2}}$	.00818121	6.28316941

It now appears that the first four decimal figures do not change as the number of sides is increased, but will remain the same *how far soever we proceed.* We may therefore consider 6.28317, as *approximately* the circumference of a circle whose radius is 1, *i. e.*, $2\pi = 6.28317$, nearly, and $\pi = 3.1416$, nearly.

373. SCH.—The symbol π is much used in mathematics, and signifies, primarily, *the semi-circumference of a circle whose radius is* 1. $\frac{1}{2}\pi$ is therefore a symbol for a quadrant, 90°, or a right angle. $\frac{1}{4}\pi$ is equivalent to 45°, etc., the radius being always supposed 1, unless statement is made to the contrary. The numerical value of π has been sought in a great variety of ways, all of which agree in the conclusion that it cannot be exactly expressed in decimal numbers, but is approximately as given in the proposition. From the time of *Archimedes* (287 B.C.) to the present, much ingenious labor has been bestowed upon this problem. The most expeditious and elegant methods of approximation are furnished by the *Calculus*. The following is the value of π extended to fifteen places of decimals: 3.141592653589793.

PROPOSITION XII.

374. Theorem.—*The circumference of any circle is* $2\pi r$, *r being the radius.*

DEM.—The circumferences of circles being to each other as their radii, and

2π being the circumference of a circle whose radius is 1, we have $2\pi : circf. :: 1 : r$, whence $circf. = 2\pi r$.

375. Cor.—*The circumference of a circle is πD, D being the diameter*, since $2\pi r = \pi 2r = \pi D$.

AREA OF THE CIRCLE.

PROPOSITION XIII.

376. Theorem.—*The area of a circle whose radius is 1, is π.*

Dem.—The area of a circle is $\frac{1}{2} r \times$ *circumference* (*328*). When $r = 1$, *circumference* $= 2\pi$ (*372*); hence

area of circle whose radius is $1 = \frac{1}{2} \times 2\pi = \pi$. Q. E. D.

PROPOSITION XIV.

377. Theorem.—*The area of any circle is πr^2, r being the radius.*

Dem.—The areas of circles being to each other as the squares of their radii, and π being the area of a circle whose radius is 1, we have

$$\pi : area\ of\ any\ circle :: 1^2 : r^2,$$

whence *area of any circle* $= \pi r^2$. Q. E. D.

378. Cor.—*The area of any sector is such a part of the area of the circle as the angle of the sector is of four right angles.*

379. Sch.—As the value of π cannot be exactly expressed in numbers, it follows that the area cannot. Finding the area of a circle has long been known as the problem of *Squaring the Circle*, *i. e.*, finding a square equal in area to a circle of given radius. Doubtless many hare-brained visionaries or ignoramuses will still continue the chase after the phantom, although it has long ago been demonstrated that the diameter of a circle and its circumference are incommensurable. It is, however, an easy matter to conceive a square of the same area as any given circle. Thus, let there be a rectangle whose base is equal to the circumference of the circle, and whose altitude is half the radius; its area is exactly equal to the area of the circle. Now, let there be a square whose side is a mean proportional between the altitude and base of this rectangle; the area of the square is exactly equal to the area of the circle.

EXERCISES.

1. Show that if a chord of a circle is conceived to revolve, varying in length as it revolves, so as to keep its extremities in the circumference while it constantly passes through a fixed point, the rectangle of its segments remains constant.

2. The two segments of a chord intersected by another chord are 6 and 4, and one segment of the other chord is 3. What is the other segment of the latter chord?

3. Show how PROP'S I., II., and III. may be considered as different cases of one and the same proposition.

SUG'S.—By stating Propositions I. and II. thus, *The distances from the intersection of the lines to their intersections with the circumference*, what follows? In *Fig.* 245, if the secant AO becomes a tangent, what does OD become?

4. In a triangle whose sides are 48, 36, and 50, where do the bisectors of the angles intersect the sides?

5. In the last example find the lengths of the bisectors.

6. Review the examples under (***111, 112, 113, 114***), and give the reasons.

7. In a circle whose radius is 20, what is the length of the arc of a sector whose angle is 30°? What is the area of this sector?

8. If a circle whose radius is 24 is divided into 5 equal parts by concentric circumferences, what are the diameters of the several circles? If the radius is r, and number of parts n?

9. ***Prob.***—*To divide a line in extreme and mean ratio; that is, so that the whole line shall be to the greater segment, as the greater segment is to the less.*

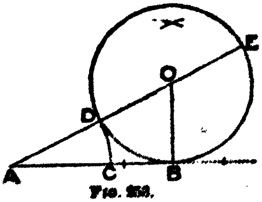


FIG. [illegible].

SOLUTION.—Let it be proposed to divide the line AB in extreme and mean ratio. At one extremity of the line, as B, erect a perpendicular equal to half the line, that is, make BO = ½ AB. With O as a centre, describe a circumference passing through B. Draw AO, and take AC equal to AD. Then is AB divided in extreme and mean ratio at C, so that AB : AC :: AC : CB. To prove it, produce AO to E. Now, AE : AB :: AB : AD (*357*), or by inversion, AB : AE :: AD : AB. By division, AB : AE — AB (= AE — DE) (or AD) (= AC) :: AD (= AC) : AB — AD (= AB — AC), (or CB). That is, AB : AC :: AC : CB.

10. ***Prob.**—To inscribe a regular decagon in a circle, and hence a regular pentagon, and regular polygons of 20, 40, 80, etc., sides.*

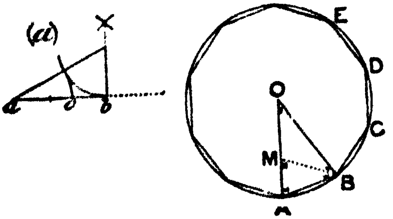


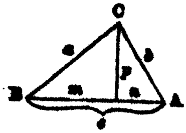
FIG. 254.

SOLUTION.—Divide the radius in extreme and mean ratio, as at (*a*). Then is the greater segment *ac* the chord of a regular inscribed decagon, as ABCD, etc. To prove this, draw OA and OB, and taking OM = *ac* = AB, a side of the polygon, draw BM. Now, OA : OM :: OM : MA by construction. As OM = AB, we have OA : AB :: AB : MA. Hence, considering the antecedents as belonging to the triangle OAB, and the consequents to the triangle BAM we observe that the two sides about the angle A, which is common to both triangles, are proportional, hence the triangles are similar (*342*). Therefore, ABM is isosceles, since OAB is, and angle BMA = A = OBA, and MB = BA = OM This makes OMB also isosceles, and the angle O = OBM. Again the exterior angle BMA = O + OBM = 2O; hence A, which equals BMA = 2O. Hence also OBA, which equals A, = 2O. Wherefore O is $\frac{1}{5}$ the sum of the angles of the triangle OAB, or $\frac{1}{5}$ of 2 right angles, = $\frac{1}{10}$ of 4 right angles. The arc AB is, therefore the measure of $\frac{1}{10}$ of 4 right angles, and is consequently $\frac{1}{10}$ of the

To construct the pentagon, join the alternate angles of the decagon. To construct the regular polygon of 20 sides, bisect the arcs subtended by the sides of the decagon, etc.

11. The projection of one line upon another in the same plane is the distance between the feet of two perpendiculars let fall from the extremities of the former upon the latter. Show that this projection is equal to the square root of the difference between the square of the line and the square of the difference of the perpendiculars.

12. In the triangle ABC, p being a perpendicular upon BA, prove that



$$m + n \ (= c) : a + b :: a - b : m - n.$$

State the fact as a proposition. Give the necessary modification when the perpendicular falls without the triangle.

SUG. $a^2 - m^2 = b^2 - n^2$, whence $a^2 - b^2 = m^2 - n^2$, etc.

13. The three sides of a triangle being 4, 5, and 6, find the segments of the last side, made by a perpendicular from the opposite angle. *Ans.* 3.75, and

14. Same as above when the sides are 10, 4, and 7, and the perpendicular is let fall from the angle included by the sides 10 and 4. Draw the figure. Why is one of the segments negative?

15. What is the area of a regular octagon inscribed in a circle whose radius is 1? What is its perimeter? What if the radius is 10?

16. What is the side of an equilateral triangle inscribed in a circle whose radius is 1?

SYNOPSIS.

GEOMETRICAL PROPERTIES DEVELOPED BY MEANS OF THE DOCTRINE OF SIMILARITY.

- Importance of this doctrine.
- RELATIONS OF THE SEGMENTS.
  - PROP. I. Of chords.
  - PROP. II. Of secants.
  - PROP. III. Of secants and tangents.
- BISECTORS OF THE ANGLES OF TRIANGLES.
  - PROP. IV. How divide sides.
  - PROP. V. Of exterior angles.
  - PROP. VI. Length of in relation to other parts.
  - PROP. VII. All intersect at one point.
- AREAS OF SIMILAR FIGURES.
  - PROP. VIII. Of triangles.
  - PROP. IX. Of polygons.
    - *Cor.* 1. As squares of diagonals.
    - *Cor.* 2. Regular polygons.
    - *Cor.* 3. As squares of apothems.
    - *Cor.* 4. Of circles.
- PERIMETERS AND RECTIFICATION.
  - Definition of rectification.
  - PROP. X. Perimeters of similar polygons.
    - *Cor.* 1. Regular polygons.
    - *Cor.* 2. Circumferences as radii.
  - PROP. XI. Rectification of circumference whose radius is 1.
    - *Sch.* Signification and importance of π.
  - PROP. XII. Circumference of any circle $= 2\pi r$.
    - *Cor.* Also πD.
- AREA OF CIRCLE.
  - PROP. XIII. Whose radius is 1.
  - PROP. XIV. Of any circle.
    - *Cor.* Of sector.
    - *Sch.* Squaring the circle.
- EXERCISES.
  - *Prob.* To divide a line in extreme and mean ratio.
  - *Prob.* To inscribe a regular decagon, etc.

CHAPTER II.

*SOLID GEOMETRY.**

SECTION I.

OF STRAIGHT LINES AND PLANES.

PERPENDICULAR AND OBLIQUE LINES.

380. *Solid Geometry* is that department of geometry in which the forms (or figures) treated are not limited to a single plane.

381. *A Plane* (or a *Plane Surface*) is a surface such that a straight line joining two points in it lies wholly in the surface.

Ill.—The surface of the blackboard is designed to be a plane. To ascertain whether it is truly so, take a ruler with a straight edge, and apply this edge in all directions upon it. If it always coincides, *i. e.*, touches throughout its whole length, the surface is a plane. Is the surface of the stove-pipe a plane? Will a straight line coincide with it in any direction? Will it in direction?

PROPOSITION I.

382. *Theorem.*—*Three points not in the same straight line determine the position of a plane.*

Dem.—Let A, B, and C be three points not in the same straight line; then one plane can be passed through them, and only one; *i. e.*, they determine the position of a plane.

Geometry of Space" is preferable to this term; but, as and as this has the advantage of simplicity and long use, the

For, pass a straight line through any two of these points, as A and B. Now, conceive any plane containing these two points; then will the line passing through them lie wholly in the plane (*381*). Conceive this plane to revolve on the line as an axis until the point C falls in the plane. Thus we have one plane passed through the three points. That there can be only one is evident, since when C falls in the plane, if it be revolved either way, C will not be in it. The same may be shown by first passing a plane through B and C, or A and C. There is, therefore, only one position of the plane in which it will contain the third point. Q. E. D.

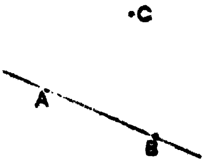


FIG. 253.

383. COR. 1.—*Through one line, or two points, an infinite number of planes can be passed.*

384. COR. 2.—*Two intersecting lines determine the position of a plane.*

DEM.—For, the point of intersection may be taken as one of the three points requisite to determine the position of a plane, and any other two points, one in each of the lines, as the other two requisite points. Now, the plane passing through these points contains the lines, for it contains two points in each line.

385. COR. 3.—*Two parallel lines determine the position of a plane.*

DEM.—For, pass a plane through one of the parallels, and conceive it revolved until it contains some point of the second parallel; then as the plane cannot be revolved either way from this position without leaving this point without it, it is the *only* plane containing the first parallel and this point in the second. But parallels lie in the same plane (*66*), whence the plane of the parallels must contain the first line, and the specified point in the second. Therefore, the plane containing the first line and a point in the second is the plane of the parallels, and is fixed in position.

386. COR. 4.—*The intersection of two planes is a straight line.*

For *two* planes cannot have even three points, *not in the same straight line*, common, much less an indefinite number, which would be required if we conceived the intersection (that is, the common points) to be any other than a straight line.

387. ***A Perpendicular to a Plane*** is a line which is perpendicular to all lines of the plane passing through its foot. Conversely, the plane is perpendicular to the line.

PROPOSITION II.

388. ***Theorem.****—A line which is perpendicular to two lines of a plane, at their intersection, is perpendicular to the plane.*

DEM.—Let PD be perpendicular to AB and CF at D; then is it perpendicular to MN, the plane of the lines AB and CF.

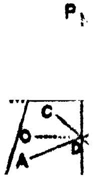


FIG. [illegible]

Let OQ be any other line of the plane MN, passing through D. Draw FB intersecting the three lines AB, CF, and OQ. Produce PD to P′, making P′D = PD, and draw PF, PE, PB, P′F, P′E, and P′B. Then is PF = P′F, and PB = P′B, since FD and BD are perpendicular to PP′, and PD = P′D (*284*). Hence, the triangles PFB and P′FB are equal (*292*); and, if PFB be revolved upon FB till P falls at P′, PE will fall in P′E. Therefore OQ has E equally distant from P and P′, and as D is also equidistant from the same points, OQ is perpendicular to PD at D (*130*). Now, as OQ is *any* line, PD is perpendicular to any line of the plane passing through its foot, and consequently perpendicular to the plane (*387*). Q. E. D.

389. COR.—*If one of two perpendiculars revolves about the other as an axis, its path is a plane perpendicular to the axis.*

Thus, if AB revolves about PP′as an axis, it describes the plane MN.

PROPOSITION III.

390. ***Theorem.****—At any point in a plane one perpendicular can be erected to the plane, and only one.*

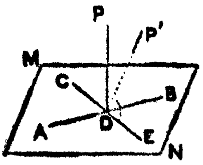


FIG. [illegible]

DEM.—Let it be required to show that one perpendicular, and only one, can be erected to the plane MN at D. Through D draw two lines of the plane, as AB and CE, at right angles to each other. CE being perpendicular to AB, let a line be conceived as starting from the position ED to revolve about AB as an axis. It will remain perpendicular to AB (*389*). Conceive it to have passed to P′D. Now, as it continues to revolve, P′DC diminishes continuously, and at the same rate as P′DE grows greater; hence, in of the revolving line, and in only one, as PD, PDE will equal PDC, and PD will be perpendicular to CE. Therefore, PD is perpendicular to two lines of the plane, at their intersection, and is the only line that can be thus perpendicular, whence it is perpendicular to the plane (*388*), and is the only perpendicular. Q. E. D.

PROPOSITION IV.

391. Theorem.*—If from any point in a perpendicular to a plane, oblique lines be drawn to the plane, those which pierce the plane at equal distances from the foot of the perpendicular are equal; and of those which pierce the plane at unequal distances from the foot of the perpendicular, those which pierce at the greater distances are the greater.*

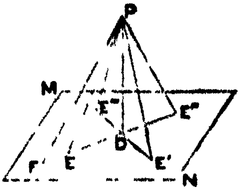


Fig. 258.

DEM.—Let PD be a perpendicular to the plane MN, and PE, PE′, PE″, and PE‴, be oblique lines piercing the plane at equal distances ED, E′D, E″D, and E‴D, from the foot of the perpendicular, then PE = PE′ = PE″ = PE‴. For each of the triangles PDE, PDE′, etc., has two sides and the included angle equal to the corresponding parts in the other.

Again, let FD be longer than E′D. Then is PF > PE′. For, take ED = E′D; then PE = PE′, by the preceding part of the demonstration. But PF > PE by (***139***). Hence, PF > PE′. Q. E. D.

392. The Inclination of a line to a plane is measured by the angle which the line makes with a line of the plane passing through the point in which the line pierces the plane and the foot of a perpendicular to the plane from any point in the line.

Thus PFD is the inclination of PF to the plane MN.

393. COR. 1.—*The angles which oblique lines drawn from a common point in a perpendicular to a plane, and piercing the plane at equal distances from the foot of the perpendicular, make with the perpendicular, are equal; and the inclinations of such lines to the plane are equal.*

Thus the equality of the triangles, as shown in the demonstration, shows that EPD = E′PD = E″PD = E‴PD, and PED = PE′D = PE″D = PE‴D.

394. COR. 2.—Conversely, *If the angles which oblique lines drawn from a point in a perpendicular to a plane, make with the perpendicular, are equal, the lines are equal, and pierce the plane at equal distances from the foot of the perpendicular.*

DEM.—Thus, in the figure, let DPE′ = DPE″; then PE′ = PE″ and DE′ = DE″. For, revolve DE′P about PD; DE′ will continue in the plane MN, and when angle DPE′ coincides with its equal DPE″, PE′ coincides with PE″, and DE′ with DE″.

395. Cor. 3.—Also, conversely, *Equal oblique lines from the same point in the perpendicular, pierce the plane at equal distances from the foot of the perpendicular.*

Dem.—Let $PE' = PE''$; then is $DE' = DE''$. For, let PDE' revolve upon PD until $E'D$ falls in $E''D$; then, if DE' were less than DE'', PE' would be less than PE''; and, if DE' were greater than DE'', PE' would be greater than PE''. But both of these conclusions are contrary to the hypothesis. Hence, as DE' can neither be less nor greater than DE'', it must equal it. This corollary follows also from (

Cor. 4.—*The perpendicular is the shortest line that can be drawn to a plane from a point without, and measures the distance of the point from the plane.*

PROPOSITION V.

397. ***Theorem.***—*From a point without a plane one perpendicular can be drawn to the plane, and only one.*

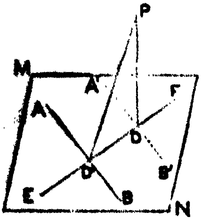


FIG. 259.

Dem.—Let it be required to show that one perpendicular can be drawn from P to the plane MN, and only one. Take AB, any line of the plane, and conceive PD' perpendicular to it. Through D' draw EF, a line of the plane, perpendicular to AB. Now, if $PD'E = PD'F$, they are both right angles, and PD' is perpendicular to two lines of the plane passing through its foot, and hence perpendicular to the plane (*388*). If, however, $PD'E$ does not equal $PD'F$, in the first instance, but $PD'F < PD'E$, conceive the line AB to move along the plane, continuing parallel to its primitive position, so as to cause D' to move towards F, thus diminishing $PD'E$ and increasing $PD'F$. At the same time observe that, if necessary in order to keep $PD'A = PD'B$*, EF can move along the plane parallel to its first position. Now, as $PD'F$ increases, passing through all successive values, and $PD'E$ diminishes in the same way, there will be some position of PD', as PD, in which $PDF = PDE$, and as by hypothesis PDA' remains $= PDB'$, PD becomes perpendicular to two lines passing through its foot, and hence perpendicular to the plane.

That there can be only one perpendicular is evident, since, if there were two, as PD' and PD, there would be two right angles in the triangle $PD'D$.

---

* According to the conception here as

398. Cor.—*Through a given point in a line one plane can be passed perpendicular to the line, and only one.*

Dem.—Let D be the point in the line PD. Pass two lines through D, as EF, and A'B', each perpendicular to PD; the plane of these lines is perpendicular to PD. Moreover, the required plane must contain both these lines, for if it passed through D and did not contain DF, there would be one line of the plane, at least, which would pass through D and not be perpendicular to PD, which is impossible. Hence, there can be no other plane than the plane of the two perpendiculars EF and A'B' which shall be perpendicular to PD, through D.

PROPOSITION VI.

399. ***Theorem.***—*If from the foot of a perpendicular to a plane a line be drawn at right angles to any line of the plane, and this intersection be joined with any point in the perpendicular, the last line is perpendicular to the line of the plane.*

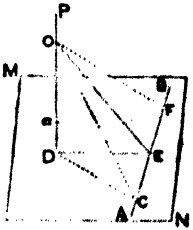


Fig. 200.

Dem.—From the foot of the perpendicular PD let DE be drawn perpendicular to AB, any line of the plane MN, and E joined with O, any point of the perpendicular; then is OE perpendicular to AB.

Take EF = EC, and draw CD, FD, CO, and FO. Now, CD = DF (?)[*], whence CO = FO (?), and OE has O equally distant from F and C, and also E. Therefore, OE is perpendicular to AB (?). Q. E. D.

400. Cor.—*The line* DE *measures the shortest distance between* PD *and* AB.

For, if from any other point in AB, as C, a line be drawn to D, it is longer than DE (?); and if drawn from C to *a*, any other point in PD than D, C*a* is longer than CD (?), and consequently longer than DE (?).

PARALLEL LINES AND PLANES.

401. ***A Line is Parallel*** to a plane when the two will not meet, how far soever they be produced. The plane is also said to be parallel to the line.

---

[*] Hereafter the reason will be often left out, and the mark (?) will be used to indicate that the student is to supply it.

PROPOSITION VII.

402. Theorem.—*One of two parallel lines is parallel to every plane containing the other.*

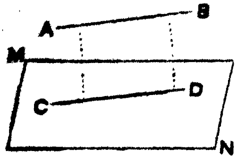


Fig. 261.

Dem.—AB being parallel to CD is parallel to any plane MN containing CD.

Since AB and CD are in the same plane (?), and as the intersection of their plane with MN is CD (?), if AB meets the plane MN, it must meet it in CD, or CD produced. But this is impossible (?). Whence AB is parallel to MN. Q. E. D.

403. Cor. 1. *Through any given line a plane may be passed parallel to any other given line not in the plane of the first.*

For, through any point of the line through which the plane is to pass, conceive a line parallel to the second given line. The plane of the two intersecting lines is parallel to the second given line (?).

404. Cor. 2.—*Through any point in space a plane may be passed parallel to any two lines in space.*

For, through the given point, conceive two lines parallel to the given lines; then is the plane of these intersecting lines parallel to the two given lines (?).

PROPOSITION VIII.

405. Theorem.—*If one of two parallels is perpendicular to a plane, the other is perpendicular also.*

Dem.—Let AB be parallel to CD, and perpendicular to the plane MN; then is CD perpendicular to MN.

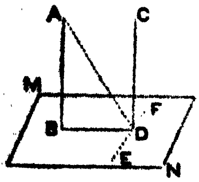


Fig. 262.

For drawing BD in the plane MN, it is perpendicular to AB (?), and consequently to CD (?). Through D draw EF in the plane and perpendicular to BD, and join D with any point in AB, as A; then is EF perpendicular to AD (?). Now, EF being perpendicular to two lines, AD and BD of the plane ABDC, is perpendicular to the plane, and hence to any line of the plane passing through D, as CD. Therefore CD is perpendicular to BD and EF, and consequently to the plane MN (?). Q. E. D.

406. Cor. 1.—*Two lines which are perpendicular to the same plane are parallel.*

Thus, AB and CD being perpendicular to the plane MN, if AB is not parallel to CD, draw a line through B which shall be. By the proposition this line is perpendicular to MN, and hence must coincide with AB ([illegible]).

407. Cor. 2.—*Two lines parallel to a third not in their own plane are parallel to each other.*

Dem.—If AB and CD are parallel to EF, they are parallel to each other. Let MN be a plane passing through EF at F′, and to which EF is perpendicular; then are AB and CD respectively perpendicular to MN (?), and hence parallel (?). Q. E. D.

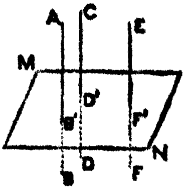


Fig. 203.

408. ***Parallel Planes*** are such as do not meet when indefinitely produced.

PROPOSITION IX.

409. ***Theorem.***—*Two planes perpendicular to the same line are parallel to each other.*

Dem.—For, if they could meet in some point, as O, conceive two lines drawn from O, one in each plane, to the points where the perpendicular pierces the planes. We should then have two lines from the same point, perpendicular to the same line (?), which is impossible. Hence, as the planes cannot meet, they are parallel. Q. E. D.

PROPOSITION X.

410. ***Theorem.***—*If a plane intersect two parallel planes, the lines of intersection are parallel.*

Dem.—Let AD intersect the parallel planes MN and PQ in AB and CD; then is AB parallel to CD. For, if AB and CD could meet, the planes MN and PQ would meet, as every point in AB is in MN, and every point in CD in PQ. Hence, AB and CD lie in the same plane, and do not meet how far soever they be produced; they are therefore parallel. Q. E. D.

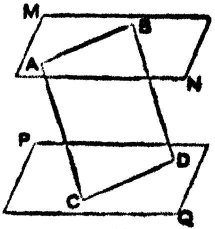


Fig. 204.

411. Cor.—*Parallel lines intercepted between parallel planes are equal.*

Thus AC = BD if they are parallel. For, the intersections AB and CD, of the plane of these parallels, are parallel (?), and the figure ABDC is a parallelogram; whence AC = BD (?).

PROPOSITION XI.

412. ***Theorem.***—*A line which is perpendicular to one of two parallel planes, is perpendicular to the other also.*

DEM.—Let AB be perpendicular to MN; then is it perpendicular to PQ, parallel to MN. For, pass two planes through AB, and let CA, DB, and EA, FB, be their intersections with the planes MN and PQ. Now CA and EA are perpendicular to AB (?); hence DB and FB being parallel to CA and EA (?) are perpendicular to AB (?). Therefore AB is perpendicular to two lines of the plane PQ, and consequently to the plane (?). Q. E. D.

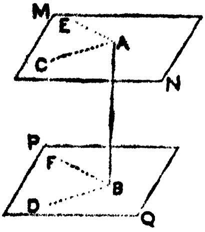


FIG. 205.

413. COR. 1.—*Through any point out of a plane, one plane can be passed parallel to the given plane, and only one.*

DEM.—To pass a plane through B parallel to MN, draw the perpendicular BA from B upon MN. Draw any two lines in the plane MN, through A, as CA and EA. Through B draw DB and FB parallel to CA and EA; then is PQ, the plane of these lines, perpendicular to AB, and hence parallel to MN. As the plane parallel to MN must contain FB and DB, and as but one plane can be passed through these lines, there can be only one plane through B parallel to MN.

414. The Distance *between two parallel planes at any point is measured by the perpendicular.*

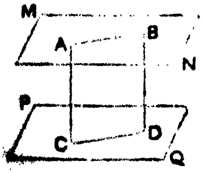


FIG. 206.

415. COR. 2.—*Parallel planes are everywhere equally distant from each other.*

DEM.—Let A and B be any two points in the plane MN, and AC and BD the perpendiculars from these points, let fall on the parallel plane PQ; then are they perpendicular to MN by the proposition, and since the figure ABCD is a parallelogram (?) [a rectangle, also (?)], AC = BD.

PROPOSITION XII.

416. Theorem.—***Two angles lying in different planes, but having their sides parallel and extending in the same direction, are equal, and their planes are parallel.***

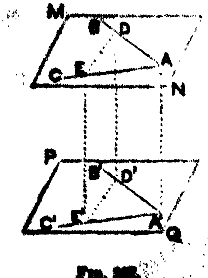


FIG. 207.

DEM.—Let A and A' lie in the different planes MN and PQ, and have AB parallel to A'B', and AC to A'C'; then A = A', and MN and PQ are parallel.

For, take AD = A'D', and AE = A'E', and draw AA', DD', EE', ED, and E'D'. Now, AD being equal and parallel to A'D', AA' = DD' (?). For like reason AA' = EE', therefore EE' = DD'. Again, since EE' and DD' are respectively parallel to AA', they are parallel to each other (?), whence EDD'E' is a parallelogram (?), and ED = E'D'. Hence the triangles ADE and A'D'E' are

mutually equilateral, and A, opposite ED, is equal to A′, opposite E′D′ equal to ED.

Again, the plane of the angle BAC, MN, is parallel to PQ, the plane of B′A′C′. For, let a plane be passed through AC and revolved until it is parallel to PQ. It must cut DD′, which is parallel to AA′, and EE′, so that DD′ shall equal AA′ and EE′ (?); hence it must pass through D.

417. Cor. 1.—*If two intersecting planes be cut by parallel planes, the angles formed by the intersections are equal.*

Thus, AB′ and AE′ being cut by the parallel planes MN and PQ, AD is parallel to A′D′ (?), and lies in the same direction, and AE to A′E′. Hence BAC = B′A′C′ (?).

418. Cor. 2.—*If the corresponding extremities of three equal parallel lines not in the same plane, be joined, the triangles formed are equal, and their planes parallel.*

Thus, if AA′ = DD′ = EE′, the sides of the triangle AED are equal to the sides of A′E′D′, since the figures AD′, DE′, and EA′ are parallelograms (?), and the corollary comes under the proposition (?).

PROPOSITION XIII.

419. *Theorem.*—*The corresponding segments of lines cut by parallel planes are proportional.*

Dem.—Let AB, CD and EF be cut by the parallel planes MN, PQ, RS, and TU; then A*a* : C*e* :: *ab* : *ef* :: *b*B : *f*D, and A*a* : E*i* :: *ab* : *ik* :: *b*B : *k*F, and C*e* : E*i* :: *ef* : *ik* :: *f*D : *k*F.

For, join the extremities A and D, and E and D, and conceive the intersections of the plane of AB and AD with the parallel planes to be BD, *bd*, and *ac*. These lines are parallel (?), and A*a* : A*c* :: *ab* : *cd* :: *b*B : *d*D (?). For a similar reason, C*e* : A*c* :: *ef* : *cd* :: *f*D : *d*D (?). Whence, the consequents of the proportions being the same, the antecedents give A*a* : C*e* :: *ab* : *ef* :: *b*B : *f*D. In like manner we can show that C*e* : E*i* :: *ef* : *ik* :: *f*D : *k*F. [Let the student give the details.] From these proportions we have A*a* : E*i* :: *ab* : *ik* :: *b*B : *k*F (?). Q. E. D.

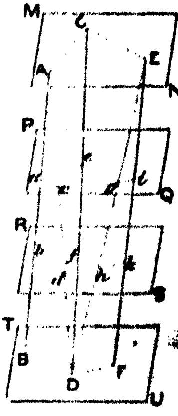


FIG. 266.

EXERCISES.

1. Designate any three points in the room, as one corner of the desk, a point on the stove, and some point in the ceiling, and show how you can conceive the plane of these points.

2. Show the position of two lines which will not meet, and yet are not parallel.

3. Conceive two lines, one line in the ceiling and one in the floor, which shall not be parallel to each other. What is the shortest distance between these lines?

4. The ceiling of my room is 10 feet above the floor. I have a 12 foot pole, by the aid of which I wish to determine a point in the floor directly under a certain point in the ceiling. How can I do it?

SUG.—Consult PROP. IV.

5. *Upon what principle in this section is it that a stool with three legs always stands firm on a level floor, when one with four may not?*

6. By the use of two carpenter's squares you can determine a perpendicular to a plane. How is it done?

7. If you wish to test the perpendicularity of a stud to a level floor, on how many sides of it is it necessary to measure the angle which it makes with the floor? By applying the right angle of the carpenter's square on *any* two sides of the stud, to test the angle which it makes with the floor, can you determine whether it is perpendicular or not?

8. We see in straight lines. If a line[*] be placed between our eye and a surface, it covers a certain space on the surface; this figure or space is said to be the *projection* of the line on that surface. Upon what principles in this section is it that the projections of straight lines are straight? Why is it that the projections of parallels which are parallel to the plane upon which we see them projected, are parallel, while parallel lines which are inclined to this plane are projected in oblique lines?

9. If a line is drawn at an inclination of 23° to a plane, what is the greatest angle which any line of the plane, drawn through the point where the inclined line pierces the plane, makes with the line? Can you conceive a line of the plane which makes an angle of 50° with the inclined line? Of 80°? Of 15°? Of 170°?

Hereafter, the student should make the synopses.

---

[*] The term line is here used in its colloquial sense, and refers to a

SECTION II.

OF SOLID ANGLES.

---

420. *A Solid Angle* is the opening between two or more planes, each of which intersects all the others. The lines of intersection are called *Edges*, and the planes, or the portion of the planes between the edges, where there are more than two, are called *Faces*.

421. *A Diedral Angle*, or simply a *Diedral*, is the opening between *two* intersecting planes.

422. *A Polyedral Angle*, called also simply a *Polyedral*, is the opening between *three* or more planes which intersect so as to have one common point, and only one. In the case of three intersecting planes the angle is called a *Triedral*. The point common to all the planes is called the *Vertex*. The plane angles enclosing a polyedral are the *Facial* angles.

423. *A Diedral (Angle) is measured* by the plane angle included by lines drawn in its faces from any point in the edge, and perpendicular thereto. A diedral angle is called right, acute, or obtuse, according as its measure is right, acute, or obtuse. Of course the magnitude of a solid angle is independent of the distances to which the edges may chance to be produced.

ILL'S.—The opening between the two planes CABF and DABE is a *Diedral* (angle). AB is the *Edge*, and CABF and DABE are the *Faces*. Let MO lie in the plane AF, perpendicular to the edge; and NO in AE, and also perpendicular to the edge; then the plane angle MON is the measure of the diedral. A diedral may be read by the letters on the edge, when there

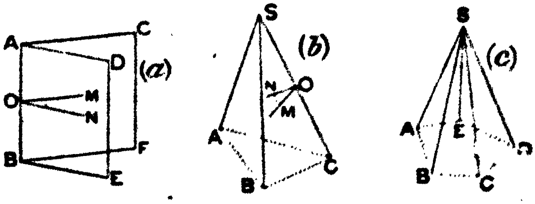


FIG. [illegible].

would be no ambiguity, or otherwise by these letters and one in each face; thus, the diedral in (*a*) may be designated as **AB**, or as **C-AB-D**.

In (*b*) we have a *Triedral* (angle). The *edges* are **SA**, **SB**, and **SC**; the *faces* **ASB**, **BSC**, and **ASC**: the *facial angles* are **ASB**, **ASC**, and **BSC**; and **S** is the *vertex*. Such an angle, and any polyedral (angle), may be read by naming the angle at the vertex, when there would be no ambiguity, or otherwise by naming the letter at the vertex, and then one in each edge; thus **S-ABCDE** designates the polyedral (*c*). The opening between the planes is the angle, in each case.

OF DIEDRALS.

424. A Diedral may be considered as generated by the revolution of a plane about a line of the plane, and hence we may see the propriety of measuring it by the angle included by two lines in its faces perpendicular to its edge, as stated in the preceding article.

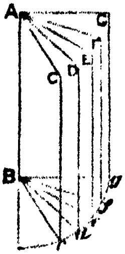


FIG. 270.

ILL.—Let **AB** be a line of the plane **CB**. Conceive *g***B** perpendicular to **AB**. Now, let the plane revolve upon **AB** as an axis, whence *g***B** describes a circle (?); and at any position of the revolving plane, as *f***BAF**, since *f***B***g* measures the amount of revolution, it may be taken as the measure of the diedral *f*-**BA**-*g*. When *g***B** has made ¼ of a revolution, the plane will have made ¼ of a revolution, and the diedral will be right.

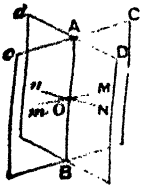


FIG. 271.

425. COR.—*Opposite diedrals are equal.*

Thus, if **C-AB-D** is measured by **MON**, *c*-**AB**-*d* is measured by the equal angle *n***O***m*.

PROPOSITION I.

426. Theorem.*—Any line in one face of a right diedral, perpendicular to its edge, is perpendicular to the other face.*

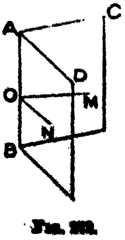


FIG. 272.

DEM.—In the face **CB** of the right diedral **C-AB-D**, let **MO** be perpendicular to the edge **AB**; then is it perpendicular to the face **DB**. For, draw **ON** in the face **DB**, and perpendicular to **AB**. Now, since the diedral is right, and **MON** measures its angle, **MON** is a right angle; whence **MO** is perpendicular to two lines of the plane **DB**, and consequently perpendicular to the plane. Q. E. D.

427. COR.—Conversely, *If one plane contain a line which is perpendicular to another plane, the diedral is right.*

Thus, if **MO** is perpendicular to the plane **DB**, **C-AB-D** is a right diedral. For **MO** is perpendicular to every line of **DB** passing through its foot (?); and hence is perpendicular to **ON**, drawn at right angles to **AB**. Whence **C-AB-D** is a right diedral, for it is measured by a right plane angle.

PROPOSITION II.

428. ***Theorem.***—*If two planes are perpendicular to a third, their intersection is perpendicular to the third plane.*

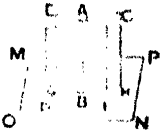


FIG. 273.

DEM.—If **CD** and **EF** are perpendicular to the plane **MN**, then is **AB** perpendicular to **MN**. For, **EF** being perpendicular to **MN**, **D-FC-E** is a right diedral, and a line in **EF** and perpendicular to **FC** at **B** is perpendicular to **MN**; also a line in the plane **CD**, and perpendicular to **DH** at **B**, is perpendicular to **MN** (?). Hence, as there can be one and only one perpendicular to **MN** at **B**, and as this perpendicular is in both planes, **CD** and **EF**, it is their intersection. Q. E. D.

PROPOSITION III.

429. ***Theorem.***—*If from any point perpendiculars be drawn to the faces of a diedral angle, their included angle will be the supplement of the angle which measures the diedral, or equal to it.*

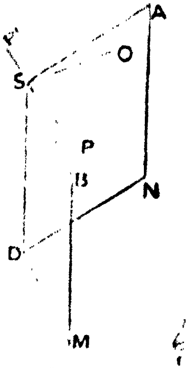


FIG. 274.

DEM.—Let **BD** and **AD** be any two planes including the diedral **A-SD-B**, then will two lines drawn from any point, perpendicular to these planes, include an angle which is the supplement of the measure of the diedral, or equal to it.

If the point from which the lines are drawn is not in the edge **SD**, we may conceive two lines drawn through any point, as **S**, in this edge, which shall be parallel to the two proposed, and hence include an equal angle, and have their plane parallel to the plane of the proposed angle (*416*). Let the latter lines be **SO** and **SP**. We are to show that **OSP** is supplemental to the measure of **A-SD-B**. A plane passed through **S**, perpendicular to the edge **SD**, will contain the lines **SO** and **SP** (*388*); and its intersections with the faces, as **SB** and **SA**, will form an angle (**ASB**) which is the measure of the diedral (*423*). Now, **PSA** = *a right angle* (?), and **OSB** = *a right angle* (?). Hence, **PSO** and **ASB** are either equal or supplemental (*283*). Q. E. D.

OF TRIEDRALS.

430. Triedrals are *Rectangular*, *Birectangular*, or *Trirectangular*, according as they have one, two, or three, right diedral angles.

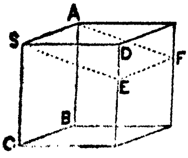


FIG. 275.

ILL'S.—The corner of a cube is a *Trirectangular* triedral, as S-ADC. Conceive the upper portion of the cube removed by the plane ASEF; then the angle at S, *i. e.*, S-AEC is a *Birectangular* triedral, A-SC-E and A-SE-C being right diedrals.

431. ***An Isosceles Triedral*** is one that has two of its facial angles equal. ***An Equilateral Triedral*** is one that has all three of its facial angles equal.

432. ***Two Symmetrical Triedrals*** are such as have the facial angles of the one equal to the facial angles of the other, each to each; but in which the equal facial angles are not similarly situated, and hence the triedrals are not necessarily capable of superposition.

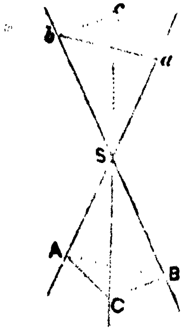


FIG. 276.

ILL'S.—Let the edges of the triedral S-ABC, be produced beyond the vertex, forming a second triedral S-*abc*; then are the two triedrals symmetrical, *i. e.*, the faces are equal plane angles, but disposed in a different order. Thus, ASB = *a*S*b*, ASC = *a*S*c*, and BSC = *b*S*c*; but the triedrals cannot be made to coincide. To show this fact, conceive the upper triedral detached, and the face *a*S*c* placed in its equal face ASC, S*a* in SA, and S*c* in SC. Now, the edge S*b*, instead of falling in SB will fall on the left of the plane ASC.

Symmetrical solids are of frequent occurrence: the two hands form an illustration; for, though the parts may be exactly alike, the hands cannot be placed so that their like parts will be similarly situated; in short, the left glove will not fit the right hand.

433. Two triedrals are ***Supplementary*** when the edges of one are perpendicular to the faces of the other. (See **438*a*.**)

PROPOSITION IV.

434. *Theorem.*—*The sum of any two facial angles of a triedral is greater than the third.*

DEM.—This proposition needs demonstration only in case of the sum of the two smaller facial angles as compared with the greatest (?). Let ASB and BSC each be less than ASC; then is ASB + BSC > ASC. For, make the angle ASb′ = ASB, and Sb′ = Sb, and pass a plane through b and b′, cutting SA and SC in a and c. The two triangles aSb and aSb′ are equal (?), whence ab′ = ab. Now, ab + bc > ac (?), and subtracting ab from the first member, and its equal ab′ from the second, we have bc > b′c. Whence the two triangles bSc and b′Sc have two sides equal, but the third side bc > than the third side b′c, and consequently angle bSc > b′Sc. Adding ASB to the former, and its equal ASb′ to the latter, we have ASB + BSC > ASC. Q. E. D.

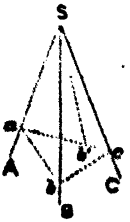


FIG. 277.

435. COR.—*The difference between any two facial angles of a triedral is less than the third facial angle* (?).

PROPOSITION V.

436. Theorem.—*The sum of the facial angles of a triedral may be anything between* 0 *and four right angles.*

DEM.—Let ASB, BSC, and ASC be the facial angles enclosing a triedral; then, as each must have some value, the sum is greater than 0, and we have only to show that ASB + ASC + BSC < 4 *right angles.* Produce either edge, as AS, to D. Now, in the triedral S-BCD, BSC < BSD + CSD. To each member of this inequality add ASB + ASC, and we have

ASB + ASC + BSC < ASB + ASC + BSD + CSD.

But, ASB + BSD = 2 right angles (?), and ASC + CSD = 2 *right angles;* whence ASB + ASC + BSD + CSD = 4 *right angles;* and consequently, ASB + ASC + BSC < 4 *right angles.* Q. E. D.

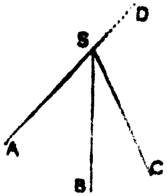


FIG. 278.

PROPOSITION VI.

437. Theorem.—*Two triedrals having the facial angles of the one equal to the facial angles of the other, each to each, and similarly arranged, are equal.*

[illegible].—In the triedrals S and *s*, let ASB = *asb*, BSC = *bsc*, and CSA = *csa*; and let these facial angles occur in the same order, then is S-ABC = *s-abc*.

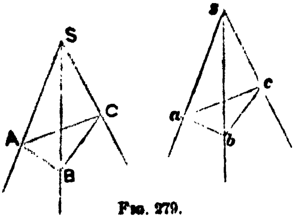


FIG. 279.

Take C, any point in SC, and make *sc*=SC, and draw AC and BC perpendicular to SC, and *ac* and *bc* perpendicular to *sc*; then ACB measures the diedral A-SC-B, and *acb* the diedral *a-sc-b*. Now the triangles SCB and *scb* are mutually equiangular (?) and have SC = *sc*; hence SB = *sb*, and CB = *cb*. For a like reason AC = *ac*, and SA = *sa*. Hence AB = *ab* (?), and the triangles ACB and *acb* are equal (?). Now, since angle ACB, ; the diedral A-SC-B, equals angle *acb*, measuring the diedral *a-sc-b*, diedrals are equal, and can be applied to each other. Applying these diedrals, since angle ASC = *asc*, and BSC = *bsc*, the edges AS and *as* coincide, as also do BS and *bs*, whence the triedrals coincide throughout, and are consequently equal. Q. E. D.

PROPOSITION VII.

438. ***Theorem.***—*Of two supplementary triedrals, the facial angles of the one are the supplements of the diedrals of the other.*

Let S-ABC be any triedral; if a second triedral be formed with its edges perpendicular to the faces of S-ABC, one to each face respectively, then are the facial angles of the one, supplements of the diedrals of the other.

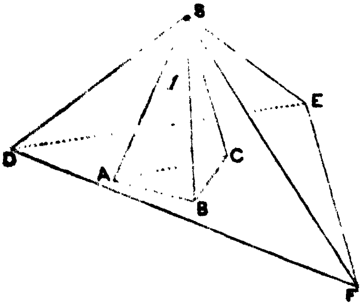


FIG. 280.

If the vertex of the second triedral is not at the vertex of the first, we may conceive a triedral formed by drawing three lines through the vertex S, as SE, SD, and SF, parallel to the edges of the given triedral; then will these edges be perpendicular to the same planes as the edges to which they are parallel (*405*), and hence the angle thus formed (S-EDF) will be a triedral supplemental to S-ABC (*433*), and the facial angles of the two having their edges parallel will be equal (*416*), and consequently the triedrals equal (*437*). Now, SE being perpendicular to ASB, and SF to ASC, angle ESF is supplemental to the diedral B-SA-C (*429*.) In like manner, SC being perpendicular to BSC, DSE is supplemental to A-SB-C, and DSF to A-SC-B.

Thus we have shown that the facial angles of of the diedrals of S-ABC. We are now to show that the facial angles of S-ABC are supplements of the diedrals of S-EDF; *i. e.*, that ASB is the supplement of D-SE-F, BSC of E-SD-F, and ASC of D-SF-E. Since SE is by hypothesis perpendicular to ASB, it is perpendicular to AS (*387*); and since SF is perpendicular to ASC, it is perpendicular to AS (*387*). Hence AS is perpendicular to the face FSE (?). In like manner we may show that SB is perpendicular to DSE, and SC to DSF; whence it follows from the preceding part of the demonstration, or directly from (*429*), that angle ASB is the supplement of D-SE-F, BSC of E-SD-F, and ASC of D-SF-E.

438a. SCH.—If any edge of S-EDF, as DS, is produced beyond S, another triedral is formed which has its edges perpendicular to the faces of S-ABC. Thus in all 4 triedrals can be formed with their edges perpendicular to the faces of S-ABC; but the proposition holds only for S-EDF.

PROPOSITION VIII.

439. Theorem.—*The sum of the diedrals of a triedral may be anything between two and six right angles.*

DEM.—Each diedral being the supplement of a plane angle of the supplementary triedral, the sum of the three diedrals is 3 times 2 right angles, or 6 *right angles – the sum of the angles of the supplementary triedral.* But this latter sum may be anything between 0 and 4 right angles (?). Hence the sum of the diedrals may be anything between 2 and 6 right angles. Q. E. D.

PROPOSITION IX.

440. Theorem.—*An isosceles triedral and its symmetrical triedral are equal.*

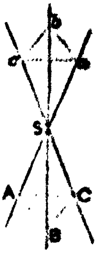


FIG. 201.

DEM.—Let S-ABC be an isosceles triedral with the facial angle ASB = BSC; then is it equal to its symmetrical triedral S-*abc*.

For, revolve S-*abc* about S until S*b* falls in SB, and bring the plane S*ba* into the plane SBC; then, since the diedrals C-SB-A and *a*-S*b*-*c* are opposite, they are equal (*425*),* and the plane S*bc* will fall in SBA. Moreover, S*a* will fall in SC, since angle BSC = ASB (by hypothesis) = *b*S*a* (vertical to ASB). In like manner S*c* will fall in SA, and the triedrals will coincide, and are therefore equal. Q. E. D.

441. SCH.—If angle ASB is not equal to BSC, it is easy to see that the ap plication will fail, notwithstanding the diedrals are equal, and the triedrals symmetrical.

---

* Should the pupil have difficulty in perceiving this, let him notice that CSB and *c*S*b* are parts of one and the same plane; and ASB and *a*S*b* are parts of another. Now *b*B is the intersection of these planes, and the diedrals mentioned are on opposite sides of this line of intersection.

442. Cor. 1.—*The diedrals opposite the equal facial angles of an isosceles triedral are equal.*

The diedral *b*-S*a*-*c* = B-SA-C being opposite, and *b*-S*a*-*c* = B-SC-A as shown in the demonstration; hence B-SA-C = B-SC-A.

443. Cor. 2.—Conversely, *If two diedrals of a triedral are equal, the triedral is isosceles.*

Dem.—If B-SA-C = B-SC-A, S-ABC is isosceles. For, revolving S-*abc* as before till the facial angle *a*S*c* falls in its equal (?) ASC, since the diedral B-SC-A = B-SA-C (by hypothesis) and the latter equals its opposite *b*-S*a*-*c*, the plane *b*S*a* will fall in the plane BSC; and, for like reasons, the plane *b*S*c* will fall in BSA. Now, as these planes coincide, their intersections S*b* and SB coincide, and the triedrals are equal; and the facial angle BSC = *b*S*a*. But *b*S*a* = ASB (?); hence ASB = BSC; *i. e.*, the triedral S-ABC is isosceles.

PROPOSITION X.

444. *Theorem.*—*Two symmetrical triedrals are equivalent.*

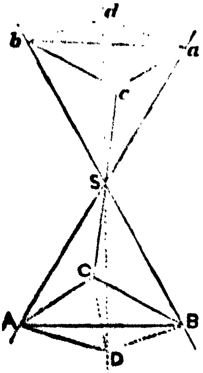


Fig. 262.

Dem.—S-ABC is equivalent to its symmetrical triedral S-*abc*.

For, let *d*D be so drawn that the angles DSA, DSC, and DSB shall be equal, and consequently *d*S*a* = *d*S*c* = *d*S*b*, and the latter respectively equal to the former. Then the isosceles symmetrical triedral S-DCB = S-*dcb*, S-DCA = S-*dca*, and S-ADB = S-*adb*. Whence the polyedral S-ABCD = S-*abcd*. Now, from the former subtracting S-ADB, and from the latter S-*adb*, there remains S-ABC = S-*abc*. Q. E. D.

445. Sch.—If *d*D fell within the given triedrals, they would be made up of the three equal triedrals, and hence equivalent.

446. *Theorem.*—*Two triedrals which have two facial angles and the included diedral equal, each to each, are equal,* e

Dem.—If the equal faces are on the same sides of the diedral in the two triedrals, the one figure can be applied to the other; and if they are on different sides, the edges of one triedral may be produced, forming the symmetrical triedral, to which the other given triedral may be applied. [Let the student construct figures, and go through with the application.]

PROPOSITION XII.

447. Theorem.—*Two triedrals which have two diedrals and the included facial angle equal, are equal, or symmetrical and equivalent.*

DEM.—[Same as in the preceding. Let the student draw figures like those for the preceding, and go through with the details of the application.]

448. COR.—*It will be observed that in equal or in symmetrical triedrals, the equal facial angles are opposite the equal diedrals.*

PROPOSITION XIII.

449. Theorem.—*Two triedrals which have two facial angles of the one equal to two facial angles of the other, each to each, and the included diedrals unequal, have the third facial angles unequal, and the greater facial angle belongs to the triedral having the greater included diedral.*

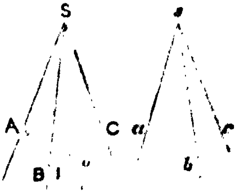


FIG. 243.

DEM.—Let **ASC** = *asc*, and **ASB** = *asb*, while the diedral **C-SA-B** > *c-sa-b*; then **CSB** > *csb*.

For, make the diedral **C-SA-*o*** = *c-sa-b*: and taking **AS*o*** = *asb*, bisect the diedral *o*-**SA-B** with the plane **ISA**. Draw *o***I** and *o***C**, and conceive the planes *o***SI** and *o***SC**. Now, the triedral **S-A*o*C** = *s-abc*, since they have two facial angles and the included diedral equal (*446*). For a like reason **S-AI*o*** = **S-AIB**, and the facial angle *o***SI** = **ISB**. Again, in the triedral **S-I*o*C**, *o***SI** + **ISC** > *o***SC** (*434*), and substituting **ISB** for *o***SI**, we have **ISB** + **ISC** (or **BSC**) > *o***SC**, or its equal *bsc*. Q. E. D.

450. COR.—Conversely, *If the two facial angles are equal, each to each, in the two triedrals, and the third facial angles unequal, the diedral opposite the greater facial angle is the greater.*

That is, if **ASB** = *asb*, and **ASC** = *asc*, while **BSC** > *bsc*, the diedral **B-AS-C** > *b-as-c*. For, if **B-AS-C** = *b-as-c*, **BSC** = *bsc* (*446*), and if **B-AS-C** < *b-as-c*, **BSC** < *bsc*, by the proposition. Therefore, as **B-AS-C** cannot be equal to nor less than *b-as-c*, it must be greater.

PROPOSITION XIV.

***451. Theorem.**—Two triedrals which have the three facial angles of the one equal to the three facial angles of the other, each to each, are equal, or symmetrical and equivalent.*

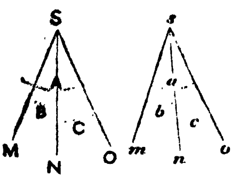


FIG. 284.

DEM.—Let A, B, and C represent the facial angles of one, and a, b, and c the corresponding facial angles of the other. If A = a, B = b, and C = c, the triedrals are equal. For A being equal to a, and B to b, if, of their included diedrals, SM were greater than sm, C would be greater than c; and if diedral SM were less than diedral sm, C would be less than c, by the last corollary. Hence, as diedral SM can neither be greater nor less than diedral sm, it must be equal to it. For like reasons, diedral SN = diedral sn, and diedral SO = diedral so. Therefore, the triedrals are equal, or symmetrical, according to the arrangement of the faces. Thus, if SN and sn are both considered as lying on the same side of the planes MSO and mso, the triedrals are equal; but, if one lies on one side and the other on the opposite side of those planes (SN in front, and sn behind, for example), the diedrals are symmetrical, and hence equivalent.

PROPOSITION XV.

***452. Theorem.**—Two triedrals which have the three diedrals of the one equal to the three diedrals of the other, each to each, are equal, or symmetrical and equivalent.*

DEM.—In the two triedrals supplementary to the given triedrals, the facial angles of the one will be equal to the facial angles of the other, each to each, since they are supplements of equal diedrals (*438*). Hence, the supplementary triedrals are equal or equivalent, by the last proposition. Now, the facial angles of the first triedrals are supplements of the diedrals of the supplementary; whence the corresponding facial angles, being the supplements of equal diedrals, are equal. Therefore, the proposed triedrals have their facial angles equal, each to each, and are consequently equal, or symmetrical and equivalent. Q. E. D.

453. COR.—*All trirectangular triedrals are equal.*

454. SCH.—The proof that two forms are equal, includes the fact that corresponding parts are equal.

OF POLYEDRALS.

455. *A Convex Polyedral* is a polyedral in which none of the faces, when produced, can enter the solid angle. A section of such a polyedral made by a plane cutting all its edges is a convex polygon. [See *Fig.* 285.]

PROPOSITION XVI.

456. *Theorem.*—*The sum of the facial angles of any convex polyedral is less than four right angles.*

DEM.—Let **S** be the vertex of any convex polyedral. Let the edges of this polyedral be cut by any plane, as **ABCDE**, which section will be a convex polygon, since the polyedral is convex. From any point within this polygon, as **O**, draw lines to its vertices, as **OA, OB, OC**, etc. There will thus be formed two sets of triangles, one with their vertices at **S**, and the other with their vertices at **O**; and there will be an equal number in each set, for the sides of the polygon form the bases of both sets. Now, the sum of the angles of one set of these triangles is equal to the sum of the angles of the other set. But the sum of the angles at the bases of the triangles having their vertices at **S** is greater than the sum of the angles at the bases of the triangles having their vertices at **O**, since $SBA + SBC > ABC$, $SCB + SCD > BCD$, etc. (*434*). Therefore the sum of the angles at **S** is less than the sum of the angles at **O**, *i. e.*, less than 4 right angles. Q. E. D.

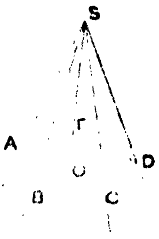


FIG. 285.

EXERCISES.

1. I have an iron block whose corners are all square (edges right diedrals, and the vertices trirectangular, or right, triedrals). If I bend a wire square around one of its edges, as $eS'd$, at what angle do I bend the wire? If I bend a wire obliquely around the edge, as aSb, at what angle can I bend it? If I bend it obliquely, as $eS''f$, at what angle can I bend it?

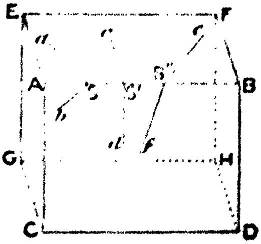


FIG. [illegible]

2. *Fig.* 286 represents the appearance of a rectangular parallelopiped, as seen from a certain position. Now, all the angles of such a solid are right angles: why is it that they nearly all *appear* oblique? Can you see a right parallelopiped from such a position that all the angles seen shall *appear* as right angles?

3. The diedral angles of crystals are measured with great care, in order to determine the substance of which the crystals consist. How must the measure be taken? If we measure obliquely around the edge, shall we get the true value of the angle?

4. Cut out any triedral from a block of wood (or a potato), and stick three pins into it, as near the vertex as you can, one in each face, and perpendicular to that face. What figure do the three pins form? What relation does the angle included between any two adjacent pins sustain to one of the diedrals of the block? Which ones are they that sustain this relation?

5. Can three planes intersect each other and yet not form a triedral angle? In how many ways? Can they all three have a common point, and yet not form a triedral?

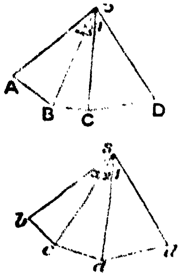


FIG. 287.

6. From a piece of pasteboard cut two figures of the same size, like ABCDS and *abcds*. Then drawing SB and SC so as to make 1 the largest angle and 3 the smallest, cut the pasteboard almost through in these lines, so that it will readily bend in them. Now fold the edges AS and DS together, and a triedral will be formed. From the piece *bcads* form a triedral in like manner, only let the lines *sc* and *sa* be drawn so as to make the angles 1, 2, and 3 of the same size as before, while they occur in the order given in *bcads*. Now, see if you can slip one triedral into the other, so that they will fit. What is the difficulty?

7. In the last case, if 1 equals $\frac{7}{8}$ of a right angle, $2 = \frac{1}{4}$ of a right angle, and $3 = \frac{3}{8}$ of a right angle, can you form the triedral? Why? If you keep increasing the size of 1, 2, and 3, until the sum becomes equal to 4 right angles, will it always be possible to form a triedral? How is it when the sum equals 4 right angles?

SECTION III.

OF PRISMS AND CYLINDERS.

457. *A Prism* is a solid, two of whose faces are equal, parallel polygons, while the other faces are parallelograms. The equal parallel polygons are the *Bases*, and the parallelograms make up the *Lateral* or *Convex Surface*. Prisms are triangular, quadrangular, pentagonal, etc., according to the number of sides of the polygon forming a base.

458. *A Right Prism* is a prism whose lateral edges are perpendicular to its bases. ***An Oblique*** prism is a prism whose lateral edges are oblique to its bases.

459. *A Regular Prism* is a right prism whose bases are regular polygons; whence its faces are equal rectangles.

460. *The Altitude* of a prism is the perpendicular distance between its bases: the altitude of a right prism is equal to any one of its lateral edges.

461. *A Truncated Prism* is a portion of a prism cut off by a plane not parallel to its base. A section of a prism made by a plane perpendicular to its lateral edges is called a *Right Section*.

ILL's.—In the figure, (*a*) and (*b*) are both prisms: (*a*) is oblique and (*b*) right. PO represents the altitude of (*a*); and any edge of (*b*), as *b*B, is its altitude. ABCDEF, and *abcdef*, are lower and upper bases, respectively. Either portion of (*b*) cut off by an oblique plane, as *a'b'c'd'e'*, is a truncated prism.

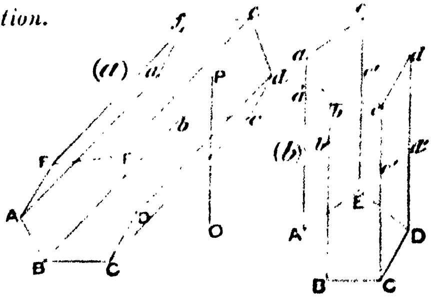


FIG. 298.

462. *A Parallelopiped* is a prism whose bases are parallelograms: its faces, inclusive of the bases, are consequently all parallelograms. If its faces are all rectangular, it is a *rectangular* parallelopiped.

463. A Cube is a rectangular parallelopiped whose faces are all equal squares.

PROPOSITION I.

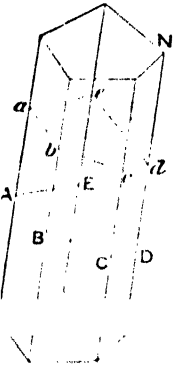


Fig. 289.

464. Theorem.*—Parallel plane sections of any prism are equal polygons.*

DEM.—Let ABCDE and *abcde* be parallel sections of the prism MN; then are they equal polygons.

For, the intersections with the lateral faces, as *ab* and AB, etc., are parallel, since they are intersections of parallel planes by a third plane (***410***). Moreover, these intersections are equal, that is, $ab = AB$, $bc = BC$, $cd = CD$, etc., since they are parallels included between parallels (***242***). Again, the corresponding angles of these polygons are equal, that is, $a = A$, $b = B$, $c = C$, etc., since their sides are parallel and lie in the same direction (***416***). Therefore the polygons ABCDE, and *abcde*, are mutually equilateral and equiangular; that is, they are equal. Q. E. D.

465. COR.—*Any plane section of a prism, parallel to its base, is equal to the base; and all right sections are equal.*

PROPOSITION II.

466. Theorem.*—If three faces including a triedral of one prism are equal respectively to three faces including a triedral of the other, and similarly placed, the prisms are equal.*

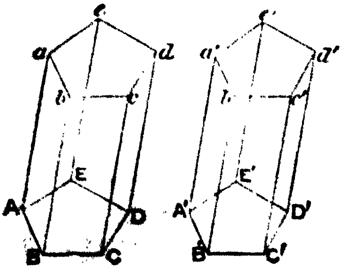


Fig. 290.

DEM.—In the prisms A*d*, and A'*d'*, let ABCDE equal A'B'C'D'E', AB*ba* = A'B'*b'a'*, and BC*cb* = B'C'*c'b'*; then are the prisms equal.

For, since the facial angles of the triedrals B and B' are equal, the triedrals are equal (***451***), and being applied they will coincide. Now, conceiving A'*d'* as applied to A*d*, with B' in B, since the bases are equal polygons, they will coincide throughout; and the faces *a*B and *a'*B' will also coincide. Whence, as *a'b'* falls in *ab*, nd $b'c'$ in bc, the upper bases, which are equal because equal to the equal lower ases, will coincide. Therefore the remaining edges will have two points common in each, and will consequently coincide.

467. Cor. 1.—*Two right prisms having equal bases and equal ltitudes are equal.*

If the faces are not similarly arranged, one prism can be inverted.

468. Cor. 2.—*The above proposition applies also to truncated risms.*

PROPOSITION III.

469. Theorem.—*Any oblique prism is equivalent to a right prism, whose bases are right sections of the oblique prism, and whose edge is equal to the edge of the oblique prism.*

Dem.—Let LB be an oblique prism, of which *abcde* and *fghil* are right sections, and *gb* = CB; then is *lb* equivalent to LB. For the truncated prisms *l*C and *e*B have the faces including any triedral, as C and B, equal and similarly placed (?), whence these prisms are equal (***466***). Now, from the whole figure, take away prism *l*C, and there remains the oblique prism LB; also, from the whole take away the prism *e*B, and there remains the right prism *lb*. Therefore, the right prism *lb* is equivalent to the oblique prism LB. Q. E. D.

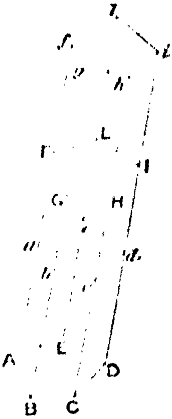


FIG. 201.

PROPOSITION IV.

470. Theorem.—*The opposite faces of a parallelopiped are equal and parallel.*

Dem.—Let A*c* be a parallelopiped, AC and *ac* being its equal bases (***462***); then are its opposite faces equal and parallel.

Since the bases are parallelograms, AB is equal and parallel to DC; and, since the faces are parallelograms, *a*A is equal and parallel to *d*D. Hence angle *a*AB = *d*DC, and their planes are parallel, since their sides are parallel and extend in the same directions. Therefore *a*B and *d*C are equal (***301***) and parallel parallelograms. In like manner it may be shown that *a*D is equal and parallel to *b*C.

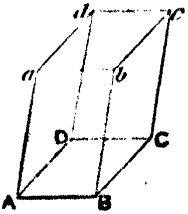


FIG. 202.

PROPOSITION V.

471. *Theorem.*—*The diagonals of a parallelopiped bisect each other.*

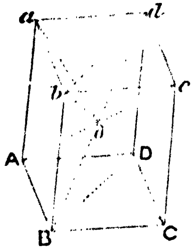


FIG. 293.

DEM.—Pass a plane through two opposite edges, as *b*B and *d*D. Since the bases are parallel, *bd* and BD will be parallel (*410*), and *b*BD*d* will be a parallelogram. Hence, *b*D and *d*B are bisected at *o* (?). For a like reason, passing a plane through *dc* and AB, we may show that *d*B and *c*A bisect each other, and hence that *c*A passes through the common centre of *d*B and *b*D. So also *a*C is bisected by *b*D, as appears from passing a plane through *ab* and DC. Hence, all the diagonals are bisected at *o*. Q. E. D.

472. COR.—*The diagonals of a rectangular parallelopiped are equal.*

PROPOSITION VI.

473. *Theorem.*—*A parallelopiped is divided into two equivalent triangular prisms by a plane passing through its diagonally opposite edges.*

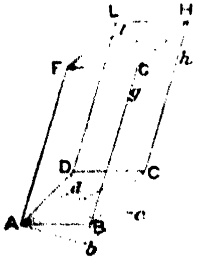


FIG. 294.

DEM.—Let H-ABCD be a parallelopiped, divided through its diagonally opposite edges FA and HC; then are the triangular prisms H-ABC, and L-ADC equivalent.

For this parallelopiped is equivalent to a right parallelopiped having a right section *Abcd* for its base, and AF for its edge (*469*), *i. e.*, H-ABCD is equivalent to *h*-*Abcd*. For the same reason the oblique triangular prism H-ABC is equivalent to the right triangular prism *h*-*Abc*; and L-ADC is equivalent to *l*-*Adc*. But *h*-*Abc* is equal to *h*-*Adc*, as they are right prisms with equal bases (*467*) and a common altitude. Hence, H-ABC is equivalent to L-ADC, as they are equivalent to two equal prisms. Q. E. D.

PROPOSITION VII.

474. *Theorem.*—***Any parallelopiped is equivalent to a rectangular parallelopiped having an equivalent base and the same altitude.***

Dem.—Let H-ABCD be any parallelopiped with all its faces oblique. 1st. By making the right section *ad*H*e*, and completing the parallelopiped *ad*H*ebc*C*f*, we have an equivalent right parallelopiped (***469***). 2d. Through the edge *ef* of this right parallelopiped make the right section *ea'b'f* and complete the parallelopiped *ea'b'f*H*d'c'*C, and we have a rectangular parallelopiped equivalent to the one previously formed (***469***), and hence equivalent to the given one. Now, the base of this rectangular parallelopiped, *i. e.*, *a'b'c'd'*, is equal to *abcd* ([illegible]), which in turn is equivalent to ABCD ([illegible]). Moreover, the altitude of the rectangular parallelopiped is the same as that of the given one, since their bases lie in the same parallel planes A*c'* and EC. Therefore, the parallelopiped H-ABCD is equivalent to the rectangular parallelopiped H-*a'b'c'd'*, which has an equivalent base and the same altitude. Q. E. D.

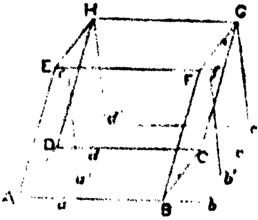


Fig. 285.

PROPOSITION VIII.

475. *Theorem.*—*The area of the lateral surface of a right prism is equal to the product of its altitude into the perimeter of its base.*

Dem.—The lateral faces are all rectangles, having for their common altitude the altitude of the prism (***460***). Whence the area of any face is the product of the altitude into the side of the base which forms its base; and the sum of the areas of the faces is the common altitude into the sum of the bases of the faces, that is, into the perimeter of the base of the prism. Q. E. D.

---

476. *A Cylindrical Surface* is a curved surface traced by a straight line moving so as to remain constantly parallel to its first position, while any point in it traces some curve. The moving line is called the *Generatrix*, and the curve traced by a point of the line is the *Directrix*.

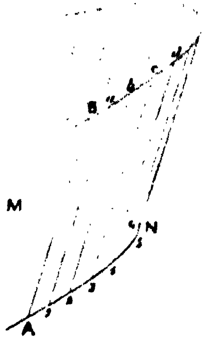


Fig. 286.

Ill.—Suppose a line to start from the position AB, and move towards N in such a manner as to remain all the time parallel to its first position **AB**, while **A** traces the curve **A** 1 2 3 4 5 6 - - - - **M**. The surface thus traced is a *Cylindrical Surface;* **AB** is the *Generatrix*, and the curve **ANM** the *Directrix.*

477. ***A Circular Cylinder***, called also a *Cylinder of Revolution*, is a solid generated by the revolution of a rectangle around one of its sides as an axis.

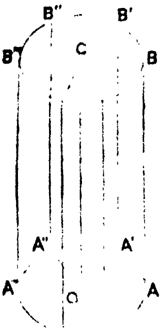


Fig. 237.

Ill.—Let **COAB** be a rectangle, and conceive it revolved about **CO** as an axis, taking successively the positions **COA'B'**, **COA"B"**, etc.; the solid generated is a *Circular Cylinder*, or a cylinder of revolution. The revolving side **AB** is the generatrix of the surface, and the circumference **OA** (or **CB**) is the directrix. This is the only cylinder treated in Elementary Geometry, and is usually meant when the word *Cylinder* is used without specifying the kind of cylinder.

478. ***The Axis*** of the cylinder is the fixed side of the rectangle. The side of the rectangle opposite the axis generates the *Convex Surface;* while the other sides of the rectangle, as **OA** and **CB**, generate the *Bases*, which in the cylinder of revolution are circles. Any line of the surface corresponding to some position of the generatrix is called an *Element* of the surface.

479. ***A Right Cylinder*** is one whose elements are perpendicular to its base. In such a cylinder any element is equal to the axis. *A Cylinder of Revolution* (**477**) is *right.*

480. A prism is said to be inscribed in a cylinder, when the bases of the prism are inscribed in the bases of the cylinder, and the edges of the prism coincide with elements of the cylinder.

PROPOSITION IX.

481. ***Theorem.***—*The area of the convex surface of a cylinder of revolution is equal to the product of its axis into the circumference of its base*, i. e., $2\pi RH$, H *being the axis and* R *the radius of the base.*

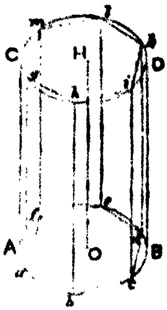


FIG. 298.

DEM.—Let a right prism, with any regular polygon for its base, be inscribed in the cylinder, as *k-abcdef*, in the cylinder whose axis is **HO**. The area of the lateral surface of the prism is **HO** (= *kb*) into the perimeter of its base, *i. e.*, **HO** × (*ab* + *bc* + *cd* + *de* + *ef* + *fa*). Now, bisect the arcs *ab*, *bc*, etc., and inscribe a regular polygon of twice the number of sides of the preceding, and on this polygon as a base construct the right inscribed prism with double the number of faces that the first had. The area of the lateral surface of this prism is **HO** × *the perimeter of its base*. In like manner conceive the operation of inscribing right prisms with regular polygonal bases continually repeated; it will *always* be true that the area of the lateral surface is equal to **HO** × *the perimeter of the base*. But the circumference of the base of the cylinder is the limit toward which the perimeters of the inscribed polygons forming the bases of the prisms constantly approach, and the convex surface of the cylinder is the limit of the lateral surface of the inscribed prism. Therefore, the area of the convex surface of the cylinder is **HO** into the circumference of the base. Finally, if R is the radius of the base, $2\pi R$ is its circumference. This multiplied by H the altitude, *i. e.*, $H \times 2\pi R$, or $2\pi RH$, is the area of the convex surface of the cylinder.

PROPOSITION X.

482. *Theorem.*—*The volume of a rectangular parallelopiped is equal to the product of the three edges of one of its triedrals.*

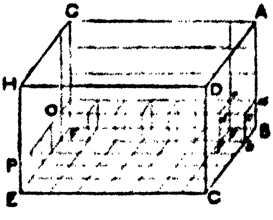


FIG. 299.

DEM.—Let **H-CBFE** be a rectangular parallelopiped. 1st. Suppose the edges commensurable, and let **BC** be 5 units in length, **BA** 4, and **BF** 7 Now conceive a cube, as *d-fb***B***g*, whose edge is one of these linear units. This cube may be used as the unit of volume. Conceive the parallelopiped **O**-*ca***B***b*, whose length is 7, and whose edges *ca* and *cb* are 1 (the linear unit of measure assumed). This parallelopiped will contain as many of the units of volume as there are linear units in **BF**: we suppose 7. Again, conceive the parallelopiped whose base is **ECBF** and altitude **PE**, one of the linear units. This parallelopiped will contain as many of the former as there are linear units in **BC**: we suppose 5. Hence this last volume is 5 × 7 = 35. Finally, there will be as many times this number of units of volume in the whole parallelopiped as AB contains linear units, or 4 × 35 = 140. Hence, when the edges are commensurable, the volume is the product of the three edges including a triedral.

2nd. When the edges are not commensurable, we reach the same conclusion by taking successively a smaller and smaller linear unit. Thus, for a first approximation take some aliquot part of one edge, as $\frac{1}{10}$ of FB. Now, by hypothesis this is not contained an exact number of times in BC, nor in BA. But conceive it as applied to BC as many times as it can be; the remainder will be less than $\frac{1}{10}$ FB. In like manner conceive it applied to AB. The volume of the parallelopiped included by these edges will be measured by the product of the edges. Now conceive the linear unit smaller. The unmeasured portion will be less. Thus, by supposing the linear unit to diminish indefinitely, we see that *it will always remain true* that the measure is the product of the three edges forming a triedral.

483. COR. 1.—*The volume of a cube is the third power of its edge.*

484. SCH.—This fact gives rise to the term *cube*, as used in arithmetic and algebra, for "third power."

485. COR. 2.—*The volume of a rectangular parallelopiped is equal to the product of its altitude into the area of its base, the linear unit being the same for the measure of all the edges.*

486. COR. 3.—*The volume of any parallelopiped is equal to the product of its altitude and the area of its base.*

For any parallelopiped is equivalent to a rectangular parallelopiped having an equivalent base and the same altitude (***474***).

487. COR. 4.—*Parallelopipeds of the same or equivalent bases are to each other as their altitudes, and those of the same altitudes are to each other as their bases. And, in general, parallelopipeds are to each other as the products of their bases and altitudes.*

PROPOSITION XI.

488.* Theorem.**—The volume of any prism is equal to the product of its altitude into its base.***

DEM.—1st. Let E-ABD be a triangular prism. Complete the parallelopiped E-ABCD. Then is E-ABD = ½ E-ABCD (*473*). But the volume of E-ABCD is equal to its altitude into its base; hence the volume of E-ABD is equal to its altitude into ½ ABCD, or ABD.

2d. Any prism may be divided into partial, triangular prisms, by passing planes through one edge and all the other non-adjacent edges, as in the figure. Let H be the altitude of the whole prism, then is it also the common altitude of the partial prisms. Now, the volume of each triangular prism is H into its base; hence, the sum of the volumes is H into the sum of the bases, *i.e.*, H into the base of the whole prism.

489. COR. 1.—*The volume of a right prism is equal to the product of its edge into its base.*

490. COR. 2.—*Prisms of the same altitude are to each other as their bases; and prisms of the same or equivalent bases are to each other as their altitudes; and, in general, prisms are to each other as the products of their bases and altitudes.*

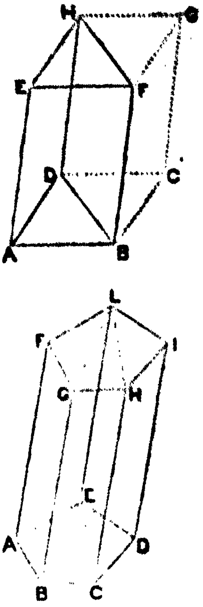


FIG. 300.

PROPOSITION XII.

491. ***Theorem.***—*The volume of a cylinder of revolution is equal to the product of its base and altitude,* i. e., πR^2H, H *being the altitude and* R *the radius of the base.*

DEM.—Inscribe any regular right prism in the cylinder, as in (*481*). The volume of this prism is equal to the product of its base and altitude; and this continues to be the fact as the number of sides of the polygon forming the base is successively doubled, and the prism approaches equality with the cylinder. Hence, as the volume of the prism is *always* equal to the product of its base and altitude, and as the altitude of the prism remains equal to the altitude of the cylinder, this fact is true when the number of the sides of the base of the prism is *infinitely* multiplied; whence the volume of the cylinder is equal to the product of its base and altitude. Now, R being the radius of the base, the area of the base is πR^2 (?): hence, the volume of the cylinder is equal to πR^2H.

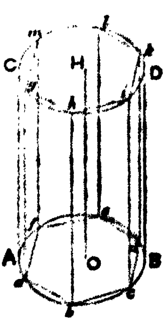


FIG. 301.

492. Cor.—*The volume of any cylinder is equal to the product of its base into its altitude.*

This can be demonstrated in a manner altogether analogous to the case given in the proposition.

---

493. ***Similar Solids*** are such as have their corresponding solid angles equal and their homologous edges proportional.

494. ***Similar Cylinders*** of revolution are such as have their altitudes in the same ratio as the radii of their bases.

495. ***Homologous Edges*** of similar solids are such as are included between equal plane angles in corresponding faces.

Ill's.—The idea of similarity in the case of solids is the same as in the case of plane figures, viz., that of *likeness of form.* Thus, one would not think such a cylinder as *one* joint of stovepipe, similar to another composed of a hundred joints of the same pipe. One would be *long* and *very slim* in proportion to its length, while the other would not be thought of as *slim.* But, if we have two cylinders the radii of whose bases are 2 and 4, and whose lengths are respectively 6 and 12, we readily recognize them as of the same shape: they are similar.

PROPOSITION XIII.

496. ***Theorem.***—*The lateral surfaces of similar right prisms are to each other as the squares of their edges (or altitudes) and as the squares of any two homologous sides of their bases,* i. e., *as the squares of any two homologous lines.*

Dem.—Let A, B, C, D, and E, be the sides of the base of one right prism whose edge (equal to its altitude) is H, and a, b, c, d, and e, the homologous sides of a similar prism whose edge is h. Letting $A + B + C + D + E = P$, and $a + b + c + d + e = p$, we have

$P : p :: A : a :: B : b :: C : c$, etc. (?).

But by hypothesis, $H : h :: A : a :: B : b$, etc.

Hence, $P : p :: H : h$. (?).

Now, $H : h :: H : h$ (?).

Whence, $P \times H : p \times h :: H^2 : h^2$ (?).

And as $H^2 : h^2 :: A^2 : a^2 :: B^2 : b^2$, etc.,

we have $P \times H : p \times h :: A^2 : a^2 :: B^2 : b^2$, etc.

But P × H is the area of the lateral surface of one prism and $p \times h$ of the other, whence the truth of the theorem appears.

PROPOSITION XIV.

497. Theorem.—*The volumes of similar prisms are to eac other as the cubes of their homologous edges, and as the cubes o their altitudes.*

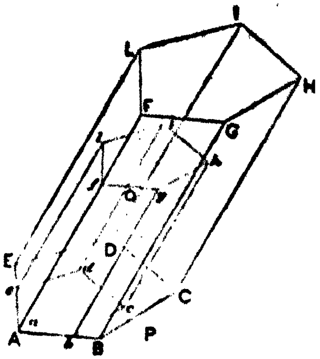


FIG. 302.

DEM.—Let H-ABCDE and *h*-*abcde* be two similar prisms, of which A and *a* are corresponding triedrals. Placing *a* so that it will coincide with A, all the faces and edges of one will be parallel to or coincident with the corresponding parts of the other, by definition (*493*). Let fall the perpendicular FP upon the common base, or its plane produced, so that FP shall equal the altitude of H-ABCDE, and OP, intercepted between the planes of the upper and lower bases of *h*-*abcde*, shall be its altitude. Call the former altitude H, and the latter *h*. Since FP and AF are cut by parallel planes, we have

$$AF : af :: H : h; \text{ and } AB : ab :: H : h,$$

since by definition AF : *af* :: AB : *ab*, etc. Call the base of H-ABCDE B, and of *h*-*abcde* *b*. Now, as the bases are similar polygons,

$$B : b :: AB^2 : ab^2 :: H^2 : h^2.$$

But $$H : h :: AB : ab :: H : h.$$

Hence, $$B \times H : b \times h :: AB^3 : ab^3 :: H^3 : h^3.$$

Now, as B × H and *b* × *h* are the volumes of the respective prisms, and a $AB^3 : ab^3$ as the cubes of any other homologous edges are to each other, the truth of the theorem is demonstrated.

PROPOSITION XV.

498. Theorem.—*The convex surfaces of similar cylinders of revolution are to each other as the squares of their altitudes, and as the squares of the radii of their bases.*

DEM.—Let H and *h* be the altitudes, and R and *r* the radii of the bases of two similar cylinders; the convex surfaces are $2\pi RH$ and $2\pi rh$ (*481*). Now,

$$2\pi RH : 2\pi rh :: RH : rh \ (?) \ (1).$$

By hypothesis, $H : h :: R : r$, or $\frac{H}{R} = \frac{h}{r}$ and $\frac{R}{H} = \frac{r}{h}$.

Multiplying the terms of the second couplet of (1) by these equals, we have,

$$2\pi RH : 2\pi rh :: H^2 : h^2,$$

and $$2\pi RH : 2\pi rh :: R^2 : r^2. \quad \text{Q. E. D.}$$

PROPOSITION XVI.

499. *Theorem.*—*The volumes of similar cylinders of revolution are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*

Dem.—Using the same notation as in the last demonstration, the student should be able to give the reasons for the following steps.

$R : r :: H : h$ (?), whence $\pi R^2 : \pi r^2 :: H^2 : h^2$ (?). Multiplying the last proportion by $H : h :: H : h$, we have $\pi R^2 H : \pi r^2 h :: H^3 : h^3$, or as $R^3 : r^3$, since $H^3 : h^3 :: R^3 : r^3$ (?). Now, $\pi R^2 H$ and $\pi r^2 h$ are the volumes of the cylinders (?); hence the volumes are to each other as the cubes of the altitudes, or as the cubes of the radii of the bases. Q. E. D.

Sch.—It is a general truth, that the surfaces of similar solids, *of any form*, are to each other as the squares of homologous lines; and their volumes are as the cubes of such lines.

EXERCISES.

1. A farmer has two grain bins which are parallelopipeds. The front of one bin is a rectangle 6 feet long by 4 high, and the front of the other a rectangle 8 feet long by 4 high. They are built between parallel walls 5 feet apart. The bottom and ends of the first, he says, are "square" (he means, it is a rectangular parallelopiped), while the bottom and ends of the other slope, *i.e.*, are oblique to the front. What are the relative capacities of the bins?

2. How many square feet of boards in the walls and bottom of the first bin mentioned in *Ex.* 1?

3. An average sized honey bee's cell is a right hexagonal prism, .8 of an inch long, with faces $\frac{3}{20}$ of an inch wide. The width of the face is always the same, but the length of the cell varies according to the space the bee has to fill. Are honey bee's cells similar? Is a honey bee's cell of the dimensions given above, similar to a wasp's cell which is 1.6 inches long, and whose face is .3 of an inch wide? How much more honey will the wasp's cell hold than the honey bee's?

4. How many square inches of sheet-iron does it take to make a joint of 7-inch stovepipe 2 feet 4 inches long, allowing an inch and a half for making the seam?

5. A certain water-pipe is 3 inches in diameter. How much water is discharged through it in 24 hours, if the current flows 3 feet per

minute? How much through a pipe of twice as great diameter, at the same rate of flow?

6. What is the ratio of the length of a hogshead holding 125 gallons, to the length of a keg of the same shape, holding 8 gallons?

7. What are the relative amounts of cloth required to clothe 3 men of the same form (similar solids), one being 5 feet high, another 5 feet 9 inches, and the other 6 feet, provided they dress in the same style? If the second of these men weighs 156 lbs., what do the others weigh?

8. If a man $5\frac{1}{2}$ feet high weighs 160 lbs., and a man 3 inches taller weighs 180 lbs., which is the stouter in proportion to his height?

9. I have a prismatic piece of timber from which I cut two blocks both 5 feet long measured along one edge of the stick; but one block is made by cutting the stick square across (a right section), and the other by cutting both ends of it obliquely, making an angle of 45° with the same face of the timber. Which block is the greater? Which has the greater lateral surface?

10. How many cubic feet in a log 12 feet long and 2 feet 5 inches in diameter? How many square feet of inch boards can be cut from such a log, allowing $\frac{1}{4}$ for waste in slabs and sawing?

SECTION IV.

OF PYRAMIDS AND CONES.

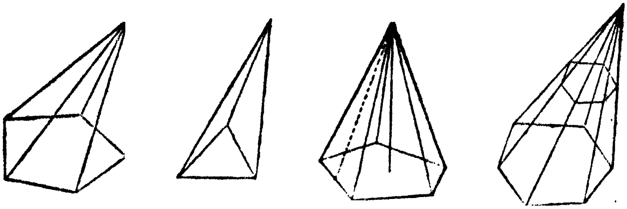
500. A Pyramid is a solid having a polygon for its base, and triangles for its lateral faces. If the base is also a triangle, it is called a triangular pyramid, or a tetraedron (*i. e.*, a solid with four faces). The vertex of the polyedral angle formed by the faces is the *vertex* of the pyramid.

501. The Altitude of a pyramid is the perpendicular distance from its vertex to the plane of its base.

502. A Right Pyramid is one whose base is a regular polygon, and the perpendicular from whose vertex falls at the middle of the base. This perpendicular is called the *axis.*

503. A Frustum of a pyramid is a portion of the pyramid intercepted between the base and a plane parallel to the base. If the cutting plane is not parallel to the base, the portion intercepted is called a *Truncated* pyramid.

504. The Slant Height of a right pyramid is the altitude of one of the triangles which form its faces. The *Slant Height of a Frustum* of a right pyramid is the portion of the slant height of the pyramid intercepted between the bases of the frustum.



ı. 303.

Ill's.—The student will be able to find illustrations of the definitions in the accompanying figures.

505. A Conical Surface is a surface traced by a line which passes through a fixed point, while any other point traces a curve. The line is the *Generatrix,* and the curve the *Directrix.* The fixed point is the *Vertex.* Any line of the surface corresponding to some position of the generatrix is called an *Element* of the surface.

506. A Cone of Revolution is a solid generated by the revolution of a right angled triangle around one of its sides, called the *Axis.* The hypotenuse describes the *Convex Surface* of the cone, and corresponds to the generatrix in the preceding definition. The other side of the triangle describes the *Base.* This cone is *right,* since the perpendicular (the axis) falls at the middle of the base. The *Slant Height* is the distance from the vertex to the circumference of the base, and is the same as the hypotenuse of the generating triangle.

507. The terms *Frustum* and *Truncated* are applied to the cone in the same manner as to the pyramid.

508. A pyramid is said to be *Inscribed* in a cone when the base of the pyramid is inscribed in the base of the cone, and the edges of the pyramid are elements of the surface of the cone. The two solids have a common vertex and a common altitude.

509. If the generatrix be considered as an indefinite straight line passing through a fixed point, the portions of the line on opposite sides of the point will each describe a conical surface. These two surfaces, which in *general* discussions are considered but one, are called ***Nappes.*** The two nappes of the same cone are evidently alike.

ILL'S.—In the figure, (*a*) represents a conical surface which has the curve ACB for its directrix, and SA for its generatrix. The figures indicate the successive positions of the point A, as it passes around the curve, while the point S remains fixed. (*b*) represents a *Cone of Revolution*, or a right cone with a circular base. It may be considered as generated in the general way, or by the right angled triangle SOA revolving about SO as an axis. SA describes the convex surface, and OA the base. The figure (*c*) represents the *Frustum of a* cone, the portion above the plane *abc* being supposed removed. Figure (*d*) represents the two nappes of an oblique cone.

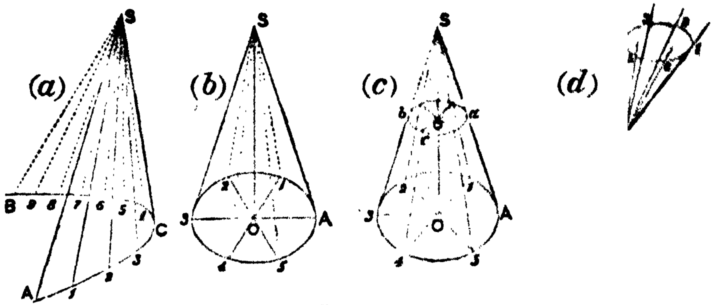


FIG. 304.

PROPOSITION I.

510. Theorem.—*Any section of a pyramid made by a plane parallel to its base is a polygon similar to the*

DEM.—The section *abcde* of the pyramid S-ABCDE, made by a plane parallel to ABCDE, is similar to ABCDE.

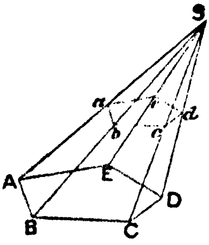


FIG. 305.

Since AB and *ab* are intersections of two parallel planes by a third plane, they are parallel (?). So also *bc* is parallel to BC, *cd* to CD, etc. Hence, angle b = B, c = C, etc. (?), and the polygons are mutually equiangular. Again, *ab* : AB :: S*b* : SB, and *bc* : BC :: S*b* : SB (?). Hence, *ab* : *bc* :: AB : BC (?). In like manner, we can show that *bc* : *cd* :: BC : CD, etc. Therefore, *abcde* and ABCDE are mutually equiangular, and have their corresponding sides proportional, and are consequently similar. Q. E. D.

PROPOSITION II.

Theorem.*—If two pyramids of the same altitude are cut by planes equally distant from and parallel to their bases, the sections are to each other as the bases.*

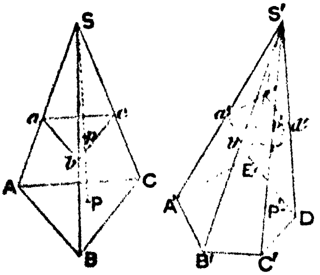


FIG. 306.

.—Let S-ABC and S′-A′B′C′D′E′ be two pyramids of the same altitude, cut by the planes *abc* and *a′b′c′d′e′*, parallel to and at equal distances from their bases; then is *abc* : *a′b′c′d′e′* :: ABC : A′B′C′D′E′.

For, conceive the bases in the same plane. Let SP = S′P′ be the common altitude, and Sp = S′$p′$ the distances of the cutting planes from the vertex. We have

$$ABC : abc :: \overline{AB}^2 : \overline{ab}^2 :: \overline{SP}^2 : \overline{Sp}^2 \text{ (?)}.$$

Also, A′B′C′D′E′ : *a′b′c′d′e′* :: $\overline{A'B'}^2 : \overline{a'b'}^2 :: \overline{S'P'}^2 : \overline{S'p'}^2$ (?).
Whence, as SP = S′P′, and Sp = S′$p′$ (?), we have

abc : *a′b′c′d′e′* :: ABC : A′B′C′D′E′ (?). Q. E. D.

512. COR.—*If the bases are equivalent, the sections are*

PROPOSITION III.

Theorem.*—The area of the lateral surface of a right pyramid is equal to the perimeter of the base multiplied by one-half*

of such a pyramid are equal isosceles triangles (?), whose altitude is the slant height of the pyramid (?). Hence, the area of these triangles is the product of one-half the slant height into the sum of their bases. But this is the lateral surface of the pyramid. (See the third cut in

514. Cor.—*The area of the lateral surface of the frustum of a right pyramid is equal to the product of its slant height into half the sum of the perimeters of its* [illegible]

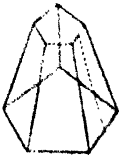


Fig. 307.

The student will be able to give the proof. It is based upon (***325***) and definitions.

PROPOSITION IV.

515. Theorem.—*The area of the convex surface of a cone of revolution (a right cone with a circular base) is equal to the product of the circumference of its base and one-half its slant height,* i. e., $\pi RH'$, R *being the radius of the base, and* H′ *the slant height.*

Dem.—In the circle which forms the base of the cone, conceive a regular polygon inscribed, as *abcde*. Joining the vertices of the angles of this polygon with the vertex of the cone, there will be constructed a right pyramid inscribed in the cone. Now, if the arcs subtended by the sides of this polygon are bisected, and these again bisected, etc., and at every step a right pyramid conceived as inscribed, it will *always* remain true that the lateral surface of the pyramid is the perimeter of its base into half its slant height. But, as the number of faces of the pyramid is increased, the perimeter of the base approaches the circumference of the base of the cone, the slant height of the pyramid approaches the slant height of the cone, and the lateral surface of the pyramid approaches the convex surface of the cone. Hence, *at* the limit we still have the same expression for the area of the convex surface, that is, the circumference of the base multiplied by half the slant height. Finally, if R is the radius of the base, its circumference is $2\pi R$, and H′ being the slant height, we have for the area of the convex surface $2\pi R \times \frac{1}{2}H'$, or

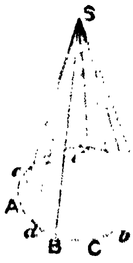


Fig. 308.

Cor. 1.—*The area of the convex surface of a cone is equal to the product of the slant height into the circumference of the circle parallel to the base, and midway between the base and vertex.*

This follows directly from the fact that the radius of the circle midway between the base and vertex is one-half the radius of the base, *i. e.*, $\frac{1}{2}R$, whence its circumference is πR. Now, $\pi R \times H'$ is the area of the convex surface, by

COR. 2.—*The area of the convex surface of the frustum of a cone is equal to the product of its slant height into half the sum of the circumferences of its bases;* i. e., $\pi\ (R + r)\ H'$, R *and* r *being the radii of its bases, and* H' *its slant height.*

From the corresponding property of the frustum of a pyramid, the student will be able to deduce the fact that $\frac{1}{2}(2\pi R + 2\pi r)\ H'$, or $\pi\ (R + r)\ H'$, is the area of this surface.

518. COR. 3.—*The area of the convex surface of the frustum of a cone is equal to the product of its slant height into the circumference of the circle midway between the bases.*

The radius of the circle midway between the bases is $\frac{1}{2}(r + R)$, whence its circumference is $\pi\ (r + R)$. Now, $\pi\ (r + R) \times H'$ is the area of the convex surface of the frustum, by the preceding corollary.

PROPOSITION V.

519. Theorem.—*Two pyramids having equivalent bases and the same altitudes are equivalent,* i. e., *equal in volume.*

DEM.—Let **S-ABCD** and **S′-A′B′C′D′E′** be two pyramids having the same altitudes, and base **ABCD** equivalent to base **A′B′C′D′E′**, *i. e.*, equal in area; then is pyramid **S-ABCD** equivalent to **S′-A′B′C′D′E′**, *i. e.*, equal in volume.

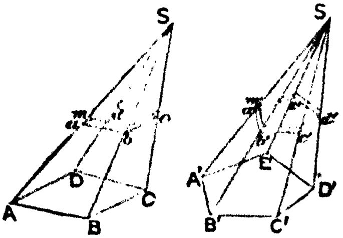


FIG. 309.

For, conceive the bases to be in the same plane, and a plane to start from coincidence with the plane of the bases, and move toward the vertices, remaining all the time parallel to the bases. At every stage of its progress the sections are equivalent, and as the plane reaches both vertices at the same time, by reason of the common altitude, it is evident that the volumes are equal.

Or, if desired, we may consider the two pyramids as divided into an equal number of infinitely thin *laminæ* parallel to the bases. Each lamina in one has its corresponding equivalent lamina in the other; hence the sum of all the *laminæ* in one equals the sum of all the *laminæ* in the other; *i. e.*, the pyramids are equivalent.

PROPOSITION VI.

520. *Theorem.*—*The volume of a triangular pyramid is equal to one-third the product of its base and altitude.*

DEM.—Let S-ABC be a triangular pyramid, whose altitude is H[*]; then is the volume equal to $\frac{1}{3}$ H × *area* ABC.

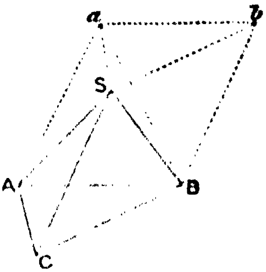


FIG. 310

For, through A and B draw A*a* and B*b* parallel to SC; and through S draw S*a* and S*b* parallel to CA and CB, and join *a* and *b*; then S*ab*-ABC is a prism with its bases equal to the base of the pyramid. Now, the solid added to the given pyramid is a quadrangular pyramid with *ab*BA as its base, and its vertex at S. Divide this into two triangular pyramids by drawing *a*B and passing a plane through SB and *a*B. These triangular pyramids are equivalent, since they have equal bases *a*AB and *ab*B and a common altitude, the vertices of both being at S. Again, S-*ab*B may be considered as having *ab*S (equal to ABC) as its base, and the altitude of the first pyramid (equal to the altitude of the prism) for its altitude, and hence as equivalent to the given pyramid. Therefore S-ABC is one third of the prism S*ab*-ABC. But the volume of the prism is H × *area* ABC. Therefore the volume of the pyramid S-ABC is $\frac{1}{3}$ H × *area* ABC. Q. E. D.

521. COR. 1.—***The volume of any pyramid is equal to one-third the product of its base and altitude.***

DEM.—Since any pyramid can be divided into triangular pyramids by passing planes through any one edge, as SE, and each of the other edges not adjacent, as SB and SC, the volume of the pyramid is equal to the sum of the volumes of several triangular pyramids having the same altitude as the given pyramid, and the sum of whose bases is the base of the given pyramid. Hence the truth of the corollary.

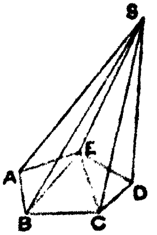


FIG. 311.

522. COR. 2.—***Pyramids having equivalent bases are to each other as their altitudes; such as have equal altitudes are to each other as their bases; and, in general, pyramids are to each other as the products of their bases and altitudes.***

---

[*] Not drawn in the figure, lest it might confuse.

PROPOSITION VII.

523. *Theorem.*—*The volume of the frustum of a triangular pyramid is equal to the volume of three pyramids of the same altitude as the frustum, and whose bases are the upper base, the lower base, and a mean proportional between the two bases of the frustum.*

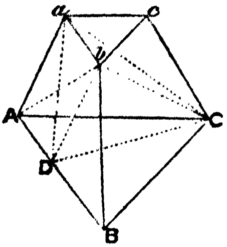


FIG. 312

DEM.—Let *abc*-ABC be the frustum of a triangular pyramid. Through *ab* and C pass a plane cutting off the pyramid C-*abc*. This has for its base the upper base of the frustum, and for its altitude the altitude of the frustum. Again, draw A*b*, and pass a plane through A*b* and *b*C, cutting off the pyramid *b*-ABC, which has the same altitude as the frustum, and for its base the lower base of the frustum. There now remains a third pyramid, *b*-AC*a*, to be examined. Through *b* draw *b*D parallel to *a*A, and draw DC and *a*D. The pyramid D-AC*a* is equivalent to *b*-AC*a*, since it has the same base and the same altitude. But the former may be considered as having ADC for its base, and the altitude of the frustum for its altitude, *i. e.*, as pyramid *a*-ADC. We are now to show that ADC is a mean proportional between *abc* and ABC.

$$ABC : abc :: \overline{AB}^2 : \overline{ab}^2 :: \overline{AB}^2 : \overline{AD}^2 \text{ (?)}.$$

Also, $ABC : ADC :: AB : AD$ (?);

whence $\overline{ABC}^2 : \overline{ADC}^2 :: \overline{AB}^2 : \overline{AD}^2$ (?).

By equality of ratios, $ABC : abc :: \overline{ABC}^2 : \overline{ADC}^2$;

whence $\overline{ADC}^2 = abc \times ABC$, *i. e.*, ADC is a mean proportional between the upper and lower bases of the frustum.

524. COR.—*The volume of the frustum of any pyramid is equal to the volume of three pyramids having the same altitude as the frustum, and for bases, the upper base, the lower base, and a mean proportional between the bases of the frustum.*

For, the frustum of any pyramid is equivalent to the corresponding frustum of a triangular pyramid of the same altitude and an equivalent base (?); and the bases of the frustum of the triangular pyramid being both equivalent to the corresponding bases of the given frustum, a mean proportional between the triangular bases is a mean proportional between their equivalents.

PROPOSITION VIII.

525. *Theorem.*—*The volume of a cone of revolution is equal to one-third the product of its base and altitude;* i. e., $\frac{1}{3}\pi R^2H$, *R being the radius of the base and H the altitude.*

DEM.—This follows from the volume of a pyramid, by a course of reasoning precisely the same as in (*515*). The volume of a pyramid being equal to one-third the product of the base and altitude, and the cone being the limit of the pyramid, the volume of the cone is one-third the product of its base and altitude. Now, R being the radius of the base of a cone of revolution, the base (area of) is πR^2, whence $\frac{1}{3}\pi R^2 H$ is the volume, H being the altitude.

526. COR. 1.—*The volume of any cone is equal to one-third the product of its base and altitude.*

527. COR. 2.—*The volume of the frustum of a cone is equal to the volume of three cones having the same altitude as the frustum, and for bases, the upper base, the lower base, and a mean proportional between the two bases of the frustum.*

The truth of this appears from the fact that the frustum of a cone is the limit of the frustum of a pyramid.

PROPOSITION IX.

528. *Theorem.*—*The lateral surfaces of similar right pyramids are to each other as the squares of their homologous edges, their slant heights, and their altitudes;* i. e., *as the squares of any two homologous dimensions.*

DEM.—Let A and a be homologous sides of the bases of two similar right pyramids, H′ and h' their slant heights, H and h their altitudes, and P and p the perimeters of their bases; then—

(1) $P : p :: A : a$, because the bases are similar polygons;
(2) $A : a :: H' : h'$, because the faces are similar triangles;
(3) $H' : h' :: H : h$ (?).

Whence, $P : p :: H' : h'$;
and, as $\frac{1}{2}H' : \frac{1}{2}h' :: H' : h'$,
multiplying, we have $\frac{1}{2}P \times H' : \frac{1}{2}p \times h' :: H'^2 : h'^2 :: A^2 : a^2 :: H^2 : h^2$. But $\frac{1}{2}P \times H'$ and $\frac{1}{2}p \times h'$ are the areas of the lateral surfaces.

PROPOSITION X.

529. *Theorem.*—*The convex surfaces of similar cones of revolution are to each other as the squares of their slant heights, the radii of their bases, and their altitudes;* i. e., *as the squares of any two homologous dimensions.*

DEM.—Let H′ and h' be the slant heights of two similar cones of revolution, R and r the radii of their bases, and H and h their altitudes; their convex surfaces are $\pi RH'$ and $\pi rh'$. Now, since the cones are similar $R : r :: H' : h'$.

Multiplying the terms of this proportion by the corresponding terms of $\pi H' : \pi h' :: H' : h'$, we have—

$$\pi RH' : \pi rh' :: H'^2 : h'^2.$$

Hence the convex surfaces are as the squares of the slant heights, and since $R : r :: H' : h' :: H : h$ (?), $R^2 : r^2 :: H'^2 : h'^2 :: H^2 : h^2$; and consequently $\pi RH' : \pi rh' :: R^2 : r^2 :: H^2 : h^2$.

PROPOSITION XI.

530. Theorem.*—The volumes of similar pyramids are to each other as the cubes of their homologous dimensions.*

Dem.—Letting A and a be homologous sides of the bases of two similar pyramids, B and b their bases, and H and h their altitudes, the student should be able to give the reasons for the following proportions :

$$B : b :: A^2 : a^2 :: H^2 : h^2.$$

$$\tfrac{1}{3}H : \tfrac{1}{3}h :: A : a :: H : h.$$

Whence $\tfrac{1}{3}BH : \tfrac{1}{3}bh :: A^3 : a^3 :: H^3 : h^3$. Q. E. D.

PROPOSITION XII.

531. Theorem.*—The volumes of similar cones are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*

Dem. R and r being the radii of their bases, and H and h their altitudes,

$R^2 : r^2 :: H^2 : h^2$ (?), and $R^3 : r^3 :: H^3 : h^3$.

Also, $\tfrac{1}{3}\pi H : \tfrac{1}{3}\pi h :: H : h$.

Multiplying, $\tfrac{1}{3}\pi R^2H : \tfrac{1}{3}\pi r^2h :: H^3 : h^3$, or as $R^3 : r^3$. Q. E. D.

EXERCISES.

1. What is the area of the lateral surface of a right hexagonal pyramid whose base is inscribed in a circle whose diameter is 20 feet, the altitude of the pyramid being 8 feet? What is the volume of this pyramid?

2. What is the area of the lateral surface of a right pentagonal pyramid whose base is inscribed in a circle whose radius is 6 yards, the slant height of the pyramid being 10 yards? What is the volume of this pyramid?

3. How many quarts will a can contain, whose entire height is 10 inches, the body being a cylinder 6 inches in diameter and 6¼ inches

high, and the top a cone? How much tin does it take to make such a can, allowing nothing for waste and the seams?

4. If very fine dry sand is piled upon a smooth horizontal surface, without any lateral support, the angle of slope (*i. e.*, the angle of inclination of the sloping side of the pile with the plane) is about 31°. Suppose two circles be drawn on the floor, one 4 feet in diameter and the other 3, and sand piles be made as large as possible on these circles as bases, no other support being given. What is the relative magnitude of the piles?

5. In the case of sand piles, as given in the last example, the ratio of the radius of the base to the altitude of the pile is $\frac{3}{5}$. How many cubic feet in each of the above piles?

6. The frustum of a right pyramid was 72 feet square at the lower base and 48 at the upper; and its altitude was 60 feet. What was the lateral surface? What the volume?

THE STUDENT SHOULD FURNISH A SYNOPSIS OF EACH SECTION AT ITS CLOSE.

SECTION V.

OF THE SPHERE[*]

532. *A Sphere* is a solid bounded by a surface every point in which is equally distant from a point within called the *Centre*. The distance from the centre to the surface is the *Radius*, and a line passing through the centre and limited by the surface is a *Diameter*. The diameter is equal to twice the radius.

---

[*] A spherical blackboard is almost indispensable in teaching this section, as well as in teaching Spherical Trigonometry. A sphere about 2 feet in diameter, mounted on a pedestal, and having its surface slated or painted as a blackboard, is what is needed. It can be obtained of the manufacturers of school apparatus, or made in any good turning-shop.

CIRCLES OF THE SPHERE.

PROPOSITION I.

533. Theorem.—*Every section of a sphere, made by a plane, is a circle.*

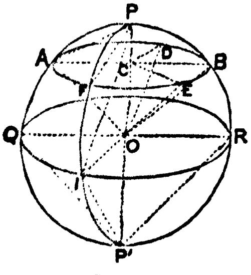


Fig. 313.

Dem.—Let AFEBD be a section of a sphere whose centre is O, made by a plane; then is it a circle.

For, let fall from the centre O a perpendicular upon the plane AFEBD, as OC, and draw CA, CD, CE, CB, etc., lines of the plane, from the foot of the perpendicular to any points in which the plane cuts the surface of the sphere. Join these points with the centre, O, of the sphere. Now, OA, OD, OB, OE, etc., being radii, are equal; whence, CA, CD, CB, CE, etc., are equal; *i. e.*, every point in the line of intersection of a plane and surface of a sphere is equally distant from a point in this plane. Hence, the intersection is a circle. Q. E. D.

534. Def.—A circle made by a plane not passing through the centre is a *Small Circle*; one made by a plane passing through the centre is a *Great Circle*.

535. Cor. 1.—*A perpendicular from the centre of a sphere, upon any small circle, pierces the circle at its centre*; and, conversely, *a perpendicular to a small circle at its centre passes through the centre of the sphere.*

536. Def.—A diameter perpendicular to any circle of a sphere is called the *Axis* of that circle. The extremities of the axis are the *Poles* of the circle.

537. Cor. 2.—*The pole of a circle is equally distant from every point in its circumference.*

The student should be able to give the reason.

538. Cor. 3.—*Every circle of a sphere has two poles, which, in case of a great circle, are equally distant from every point in the circumference of the circle; but, in case of a small circle, one pole is nearer any point in the circumference than the other pole is.*

539. COR. 4.—*A small circle is less as its distance from the centre of the sphere is greater.*

For, its diameter, being a chord of a great circle, is less as it is farther from the centre of the great circle, which is also the centre of the sphere.

540. COR. 5.—*All great circles of the same sphere are equal,* their radii being the radius of the sphere.

PROPOSITION II.

541. Theorem.—*Any great circle divides the sphere into two equal parts called Hemispheres.*

DEM.—Conceive a sphere as divided by a great circle, *i. e.*, by a plane passing through its centre, and let the great circle be considered as the base of each portion. These bases being equal, reverse one of the portions and conceive its base placed in the base of the other, the convex surfaces being on the same side of the common base. Since the bases are equal circles, they will coincide, and since every point in the convex surface of each portion is equally distant from the centre of the common base, the convex surfaces will coincide. Therefore, the portions coincide throughout, and are consequently equal. Q. E. D.

PROPOSITION III.

542. Theorem.—*The intersection of any two great circles of a sphere is a diameter of the sphere.*

DEM.—The intersection of two planes is a straight line; and in the case of the two great circles, as they both pass through the centre of the sphere, this is one point of their intersection. Hence, the intersection of two great circles of a sphere is a straight line which passes through the centre. Q. E. D.

543. COR.—*The intersections on the surface of a sphere of two circumferences of great circles are a semi-circumference, or 180°, apart, since they are at opposite extremities of a diameter.*

DISTANCES ON THE SURFACE OF A SPHERE.

544. Distances on the surface of a sphere are always to be understood as measured on the arc of a great circle, unless it is otherwise stated.

PROPOSITION IV.

545. Theorem.—*The distances, measured on the surface of a sphere, from a pole to all points in the circumference of a circle of which it is the pole, are equal.*

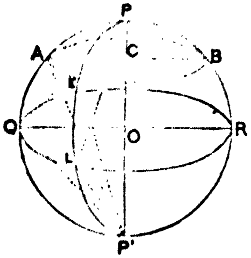


FIG. 314.

DEM.—Let **P** be a pole of the small circle **AEB**; then are the arcs **PA**, **PE**, **PB**, etc., which measure the distances on the surface of the sphere, from **P** to any points in the circumference of circle **AEB**, equal. For, by (*537*), the straight lines **AP**, **PE**, **PB**, etc., are equal, and these equal chords subtend equal arcs, as arc **PA**, arc **PE**, arc **PB**, etc., the great circles of which these lines are chords and arcs being equal (*540*). Thus, for like reasons, are **P'QA** = arc **P'LE** = arc **P'RB**, etc.

546. COR.—*The distance from the pole of a great circle to any point in the circumference of the circle is a quadrant (a quarter of a circumference).*

Since the poles are 180° apart (being the extremities of a diameter), **PAQP'** = **PELP'** = a semicircumference. But, in case of a great circle, chord **PL** = chord **P'L** (= chord **PQ** = chord **P'Q**), whence arc **PEL** = arc **P'L** = arc **PAQ** = arc **P'Q**. Hence, each of these arcs is a quadrant.

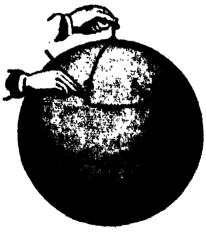


FIG. 315.

547. SCH.—By means of the facts demonstrated in this proposition and corollary, we are enabled to draw arcs of small and great circles, in the surface of a sphere, with nearly the same facility as we draw arcs and lines in a plane. Thus, to draw the small circle **AEB**, we take an arc equal to **PE**, and placing one end of it at **P**, cause a pencil held at the other end to trace the arc **AEB**, etc. To describe the circumference of a great circle, a quadrant must be used for the arc. By bending a wire into an arc of the circle, and making a loop in each end, a wooden pin can be put through one loop and a crayon through the other, and an arc drawn as represented in the figure.

PROPOSITION V.

548. Problem.—*To pass a circumference of a great circle through any two points on the surface of a sphere.*

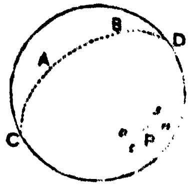


FIG. 318.

SOLUTION.—Let A and B be two points on the surface of a sphere, through which it is proposed to pass a circumference of a great circle. From B as a pole, with an arc equal to a quadrant, strike an arc *on*, as nearly where the pole of the circle passing through A and B lies, as may be determined by inspection. Then, from A, with the same arc, strike an arc *st* intersecting *on* at P. Now, P is the pole of the great circle passing through A and B. Hence, from P as a pole, with a quadrant arc draw a circle; it will pass through A and B, and will be a great circle, since its pole is a quadrant's distance from its circumference. [The student should make the construction on the spherical blackboard.]

549. COR. 1.—*Through any two points on the surface of a sphere, one great circle* can always be made to pass, and only one, except when the two points are at the extremities of the same diameter, in which case an infinite number of great circles can be passed through the two points.*

Since the arcs *on* and *st* are arcs of great circles, the circumferences of which they form parts will intersect also on the opposite side of the sphere, at a distance of a semicircumference from P. But these two points are poles of the *same* great circle. Now, as the two great circles can intersect at no other points, there can be only one great circle passed through A and B. But if the two given points were at the extremities of the same diameter, as at D and C, the arcs *st* and *on* would coincide, and *any* point in this circumference being taken as a pole, great circles can be drawn through D and C. [The student should trace the work on the spherical blackboard.]

550. SCH.—The truth of the corollary is also evident from the fact that three points not in the same straight line determine the position of a plane. Thus A, B, and the centre of the sphere, fix the position of one, and only one, great circle passing through A and B. Moreover, if the two given points are at the extremities of the same diameter, they are in the same straight line with the centre of the sphere, whence an infinite number of planes can be passed through them and the centre. The meridians on the earth's surface afford an example, the poles (of the equator) being the given points.

551. COR. 2.—*If two points in the circumference of a great circle of a sphere, not at the extremities of the same diameter, are at a quadrant's distance from a point on the surface, that point is the pole of the circle.*

---

* The word circle may be understood to refer either to the circle proper, or to its circumference. The word is in constant use in the higher mathematics, in the latter sense.

PROPOSITION VI.

552. Theorem.—*The shortest distance on the surface of a sphere, between any two points in that surface, is measured on the arc less than a semicircumference of the great circle which joins them.*

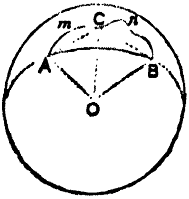


FIG. 317.

DEM.—Let A and B be any two points in the surface of a sphere, AB the arc of a great circle joining them, and $AmCnB$ any other path in the surface between A and B; then is arc AB less than $AmCnB$.

Let C be any point in $AmCnB$, and pass the arcs of great circles through A and C, and B and C. Join A, B, and C with the centre of the sphere. The angles AOB, AOC, and COB form the facial angles of a triedral, of which angles the arcs AB, AC, and CB are the measures. Now, angle $AOB < AOC + COB$ (*434*); whence arc $AB <$ arc $AC +$ arc CB, and the path from A to B is less on arc AB than on arcs AC, CB. In like manner, joining any point in AmC with A and C by arcs of great circles, their sum would be greater than AC. So, also, joining any point in CnB with C and B, the sum of the arcs would be greater than CB. As this process is indefinitely repeated, the path from A to B on the arcs of the great circles will continually increase, and also continually approximate the path $AmCnB$. Hence, arc AB is less than the path $AmCnB$. Q. E. D.

553. COR.—*The least arc of a circle of a sphere joining any two points in the surface, is the arc less than a semicircumference of the great circle passing through the points; and the greatest arc is the circumference minus this least arc.*

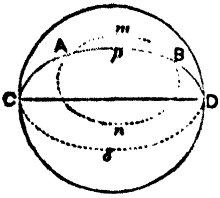


FIG. 318.

DEM.—Let $AmBn$ be any small circle passing through A and B, and $ABDoC$ the great circle. As shown above, $ApB < AmB$. Now, circumference $ABDoC >$ circumference $AmBn$ (*539*). Subtracting the former inequality from the latter, we have $BDoCA > BnA$. Q. E. D.

PROPOSITION VII.

554. Theorem.—*The shortest path on the surface of a hemisphere, from any point therein to the circumference of the great circle forming its base, is the arc less than a quadrant of a great circle perpendicular to the base, and the longest path, on any arc of a great circle, is the supplement of this shortest path.*

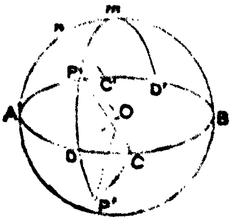


FIG. 319.

DEM.—Let P be a point in the surface of the hemisphere whose base is ACBC', and DPmD' an arc of a great circle passing through P and perpendicular to ADCBC'; then is PD the shortest path on the surface from P to circumference ADBC', and PmD' is the longest path from P to the circumference, measured on the arc of a great circle.

For, the shortest path from P to any point in circumference ADBC' is measured on the arc of a great circle (*552*). Now, let PC be any oblique arc of a great circle. We will show that arc PD < arc PC. Produce PD until DP' = PD; and pass a great circle through P' and C. Draw the radii OP, OD, OC, and OP'. The triedrals O-PDC and O-P'DC have the facial angle POD = P'OD, they being measured by equal arcs, and the facial angle DOC common. Hence, as the included diedrals are equal, both being right, the triedrals are equal or symmetrical (*446*). In this case they are symmetrical, and the facial angle POC = P'OC; whence the arc PC = arc P'C. Finally, since PC + P'C > PP', PC, the half of PC + P'C, is greater than PD, the half of PP'.

Secondly, PmD' is the supplement of PD, and we are to show that it is greater than any other arc of a great circle from P to the circumference ADBC'. Let PnC' be any arc of a great circle oblique to ADCBC'. Produce C'nP to C. Now CPnC' is a semicircumference and consequently equal to DPmD'. But we have before shown that PD < PC, and subtracting these from the equals CPnC' and DPmD', we have PmD' > PnC'.

555. COR.—*From any point in the surface of a hemisphere there are two perpendiculars to the circumference of the great circle which forms the base of the hemisphere; one of which perpendiculars measures the least distance to that circumference, and the other the greatest, on the arc of any great circle of the sphere.*

Thus PD and PmD' are two perpendiculars from P upon the circumference ADBC'.

SPHERICAL ANGLES.

556. **The angle formed by two arcs of circles of a sphere is conceived as the same as the angle included by the tangents to the arcs at the common point.**

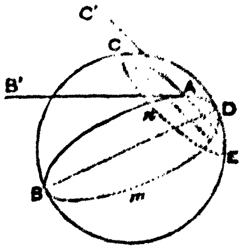


FIG. 320.

ILL.—Let AB and AC be two arcs of circles of the sphere, meeting at A; then the angle BAC is conceived as the same as the angle B'AC', B'A being tangent to the circle BADm, and C'A to the circle CAEn.

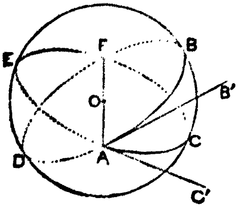


FIG. 321.

557. A Spherical Angle is the angle included by two arcs of *great circles.*

ILL.—**BAC**, *Fig.* 321, is a spherical angle, and is conceived as the same as the angle **B'AC'**, **B'A** and **C'A** being tangents to the *great circles* **BADF** and **CAEF**. [The student should not confound such an angle as **BAC**, *Fig.* 320, with a *spherical angle.*]

PROPOSITION VIII.

558. Theorem.—*A spherical angle is equal to the measure of the diedral included by the great circles whose arcs form the sides of the angle.*

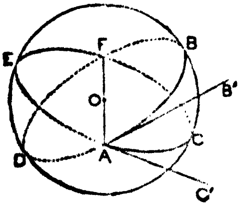


FIG. 322.

DEM.—Let **BAC** be any spherical angle, and **BADF** and **CAEF** the great circles whose arcs **BA** and **CA** include the angle; then is **BAC** equal to the measure of the diedral **C-AF-B**. For, since two great circles intersect in a diameter (*542*), **AF** is a diameter. Now **B'A** is a tangent to the circle **BADF**, that is, it lies in the same plane and is perpendicular to **AO** at **A**. In like manner **C'A** lies in the plane **CAEF** and is perpendicular to **AO**. Hence **B'AC'** is the measure of the diedral **C-AF-B** (*425*). Therefore the spherical angle **BAC**, which is the same as the plane angle **B'AC'**, is equal to the measure of the diedral **C-AF-B**. Q. E. D.

559. COR. 1.—*If one of two great circles passes through the pole of the other, their circumferences intersect at right angles.*

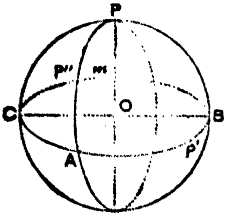


FIG. 323.

DEM.—Thus, **P** being the pole of the great circle **CAB***m*, **PO** is its axis, and any plane passing through **PO** is perpendicular to the plane **CAB***m* (*427*). Hence, the diedral **B-AO-P** is right, and the spherical angle **PAB**, which is equal to the measure of the diedral, is also right.

560. COR. 2.—*A spherical angle is measured by the arc of a great circle intercepted between its sides, and at a quadrant's distance from its vertex.*

Thus, the spherical angle **CPA** is measured by **CA**, **PC** and **PA** being quadrants. For, since **PC** is a quadrant, **CO** is perpendicular to **PO**, the edge of the diedral **C-PO-A**, and for a like reason **AO** is perpendicular to **PO**. Hence, **COA** is the measure of the diedral, and consequently **CA**, its measure, is the measure of the spherical angle **CPA**.

561. Cor. 3.—*The angle included by two arcs of small circles is the same as the angle included by two arcs of great circles passing through the vertex and having the same tangents.*

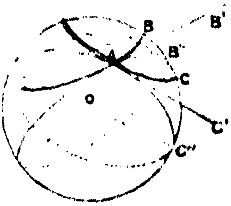


FIG. 324

Thus BAC = B''AC''. For the angle BAC is, by definition, the same as B'AC', B'A and C'A being tangents to BA and CA. Now, passing planes through C'A, B'A, and the centre of the sphere, we have the arcs B''A, C''A, and B'A, C'A tangents to them. Hence, B''AC'' is the same as B'AC', and consequently the same as BAC.

562. Sch.—*To draw an arc of a great circle which shall be perpendicular to another; or, what is the same thing, to construct a right spherical angle* Let it be required to erect an arc of a great circle perpendicular to CAB at A, *Fig.* 323. Lay off from A, on the arc CAB, a quadrant's distance, as AP', and from P' as a pole, with a quadrant describe an arc passing through A. This will be the perpendicular required.

In a similar manner we may let fall a perpendicular from any point in the surface, upon any arc of a great circle. To let fall a perpendicular from P'' upon the arc CAB, from P'' as a pole, with a quadrant describe an arc cutting CAB, as at P'. Then from P' as a pole, with a quadrant describe an arc passing through P'' and cutting CAB, and it will be perpendicular to CAB. [The student should have practice in making these constructions on the sphere.]

PROPOSITION IX.

563. Problem.—*To pass the circumference of a small circle through any three points on the surface of a sphere.*

Solution.—Let A, B, and C be the three points in the surface of the sphere through which we propose to pass the circumference of a circle. Pass arcs of great circles through the points, forming the spherical triangle ABC. Thus, to pass an arc of a great circle through B and C, from B as a pole, with a quadrant strike an arc as near as may be to the pole of the required circle; and from C as a pole, with the quadrant strike an arc intersecting the former, as at P; then is P the pole of a great circle passing through B and C (?). Hence, from P as a pole, with a quadrant pass an arc through B and C, and it will be the arc required (*551*). In like manner pass arcs through A and C, A and B. Now, bisect two of these arcs, as BC and AC, by arcs of great

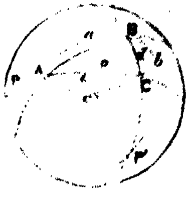


FIG. 325.

circles perpendicular to each. [The student will readily perceive how this is done.] The intersection of these perpendiculars, *o*, will be the pole of the small circle required (?). Then from *o*, as a pole, with an arc *o*B draw the circumference of a small circle: it will pass through A, B, and C (?), and hence is the circumference required.

OF TANGENT PLANES.

64. ***A Tangent Plane*** to a curved surface at a given point is the plane of two lines respectively tangent to two plane sections through the point.

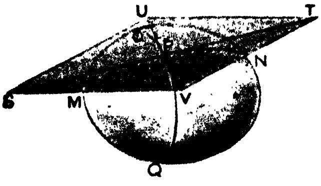


FIG. 326.

ILL.—Let P be a point in the curved surface at which we wish a tangent plane. Pass any two planes through the surface and the point P, and let OPQ and MPN represent the intersections of these planes with the curved surface. Draw UV and ST in the planes of the sections, and tangent to OPQ and MPN, at P. Then is the plane of UV and ST the tangent plane at P.

PROPOSITION X.

565. ***Theorem.****—A tangent plane to a sphere is perpendicular to the radius at the point of tangency.*

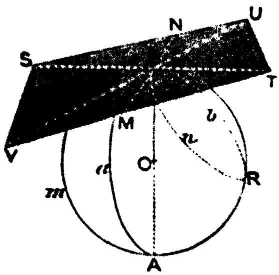


FIG. 327.

DEM.—Let P be any point in the surface of a sphere; pass two great circles, as P*a*A, etc., and P*m*AR, through P, and draw ST tangent to the arc *m*P, and UV tangent to the arc *a*P; then is the plane SVTU a tangent plane at P, and perpendicular to the radius OP. For, a tangent (as ST) to the arc *m*P is perpendicular to the radius of the circle, *i. e.*, to OP, and also a tangent (as VU) to the arc *a*P is perpendicular to the radius of *this* circle, *i. e.*, to OP. Hence, OP is perpendicular to two lines of the plane SVTU, and consequently to the plane of these lines (?). Q. E. D.

COR. 1.—*Every point in a tangent plane to a the point of tangency, is without the sphere.*

For, **OP**, the perpendicular, is shorter than any line which can be drawn from **O** to any other point in the plane (?), hence any other point in the plane than **P** lies farther from the centre of the sphere than the length of the radius, and is, therefore, without the sphere.

567. COR. 2.—*A tangent through* **P** *to* ANY *circle of the sphere passing through this point, lies in the tangent plane.*

DEM.—Thus **MN**, tangent to the small circle **P*n*R*b*** through **P**, lies in the tangent plane. For, conceive the plane of the small circle extended till it intersects the tangent plane. This intersection is tangent to the small circle, since it touches it at one point, but cannot cut it; otherwise the tangent plane would have another point than **P** common with the surface of the sphere. But there can be only one tangent to a circle at a given point. Hence this intersection is **MN**, which is consequently in the tangent plane.

OF SPHERICAL TRIANGLES.

568. A Spherical Triangle is a portion of the surface of a sphere bounded by three arcs of great circles. In the present treatise these arcs will be considered as each less than a semicircumference.

The terms scalene, isosceles, equilateral, right angled, and oblique angled, are applied to spherical triangles in the same manner as to plane triangles.

PROPOSITION XI.

569. Theorem.—*The sum of any two sides of a spherical triangle is greater than the third side, and their difference is less than the third side.*

DEM.—Let **ABC** be any spherical triangle; then is $BC < BA + AC$, and $BC - AC < BA$; and the same is true of the sides in any order. For, join the vertices **A**, **B**, and **C**, with the centre of the sphere, by drawing **AO**, **BO**, and **CO**. There is thus formed a triedral **O-ABC**, whose facial angles are measured by the sides of the triangle (***208***). Now, angle $BOC < BOA + AOC$ (***434***), whence $BC < BA + AC$: and subtracting **AC** from both members, we have $BC - AC < BA$.

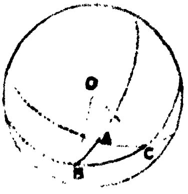


Fig. 298.

PROPOSITION XII.

570. Theorem.*—The sum of the sides of a spherical triangle may be anything between 0 and a circumference.*

DEM.—The sides of a spherical triangle are measures of the facial angles of a triedral whose vertex is at the centre of the sphere. Hence their sum may be anything between 0 and the measure of 4 right angles, as these are the limits of the sum of the facial angles of a triedral (*436*).

571. SCH.—As the sides of a spherical triangle are arcs, they can be measured in degrees. Hence, we speak of the side of a spherical triangle as 30°, 57°, 115° 10′, etc. In accordance with this, we say that the limit of the sum of the sides of a spherical triangle is 360°.

PROPOSITION XIII.

572. Theorem.*—The sum of the angles of a spherical triangle may be anything between two and six right angles.*

DEM.—The sum of the angles of a spherical triangle is the same as the sum of the measures of the diedrals of a triedral having its vertex at the centre of the sphere, as in (*569*). Now the limits of the sum of the measures of these diedrals are 2 and 6 right angles (*439*). Hence the sum of the angles of any spherical triangle may be anything between 2 and 6 right angles. Q. E. D.

573. SCH.—It will be observed, that the sum of the angles of a spherical triangle is not constant, as is the sum of the angles of a plane triangle. Thus, the sum of the angles of a spherical triangle may be 200°, 290°, 350°, 500°, anything between 180° and 540°.

574. DEF.—***Spherical Excess*** is the amount by which the sum of the angles of a spherical triangle exceeds the sum of the angles of a plane triangle; *i. e.*, it is the sum of the spherical angles −180°, or π.

ILL.—It is not difficult to observe the occasion of this *excess* in the case of the equilateral spherical triangle. Thus, let ABC be such a triangle. Conceive the plane triangle formed by the chords AB, AC, and CB. The sum of the angles of this plane triangle is 180°. But each angle of the spherical triangle is larger than the corresponding angle of the plane triangle. Thus, the spherical angle BAC is the same as the plane angle C′AB′, included between the tangents C′A and B′A, which are perpendicular to the edge of the diedral C-AO-B, and include its measuring angle. Now, CA and BA

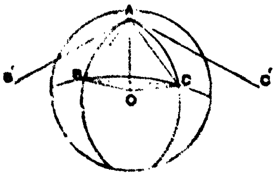


FIG. 322.

being different lines from C'A and B'A are oblique to the edge AO, and include an angle *less* than its measure, and consequently less than CAB. For a like reason the plane angle ACB < the spherical angle ACB, and plane angle ABC < spherical angle ABC. Moreover, it is easy to see that the inequality between any plane angle and the corresponding spherical angle increases as the chords BA and CA deviate more from the tangents. Whence we see why the sum of the angles of the spherical triangle is not a fixed quantity.

575. COR.—*A spherical triangle may have one, two, or even three right angles; and, in fact, it may have one, two, or three obtuse angles; since, in the latter case, the sum of the angles will not necessarily be greater than 540°.*

576. DEF.—***A Trirectangular Spherical Triangle*** is a spherical triangle which has three right angles.

PROPOSITION XIV.

577. Theorem.—*The trirectangular triangle is one-eighth of the surface of a sphere.*

DEM.—Pass three planes through the centre of a sphere, respectively perpendicular to each other. They will divide the surface into 8 trirectangular triangles, any one of which may be applied to any other. Thus, let ABA'B', ACA'C', and CBC'B' be the great circles formed by the three planes, mutually perpendicular to each other. The planes being perpendicular to each other the diedrals, as A-CO-B, C-BO-A, C-AO-B, etc., are right, and hence the angles of the 8 triangles formed are all right. Also, as AOB is a right angle, AB is a quadrant; as BOC is a right angle, CB is a quadrant, etc. Hence, each side of every triangle is a quadrant. Now any one triangle may be applied to any other. [Let the student make the application.] Hence the trirectangular triangle is one-eighth of the surface of a sphere. Q. E. D.

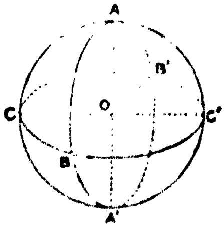


FIG. 330.

578. COR.—*The trirectangular triangle is equilateral and its sides are quadrants.*

PROPOSITION XV.

579. Theorem.—*In an isosceles spherical triangle the angles opposite the equal sides are equal;* and, conversely, *If two angles of a spherical triangle are equal, the triangle is isosceles.*

Dem.—Let ABC be an isosceles spherical triangle in which AB = AC; then angle ABC = ACB. For, draw the radii AO, CO, and BO, forming the edges of the triedral O-ABC. Now, since AB = AC, the facial angles AOC and AOB are equal, and the triedral is isosceles. Hence the diedrals A-OB-C and A-OC-B are equal (*442*), and consequently the spherical angles ABC and ACB are equal (*558*). Again, if angle ABC = angle ACB, side AC = side AB. For in the triedral O-ABC, the diedrals A-OB-C and A-OC-B are equal, whence the facial angles AOB and AOC are equal (*443*), and consequently the sides AB and AC which measure these angles.

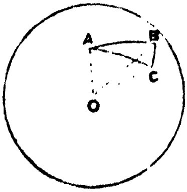


FIG. 331.

580. Cor.—*An equilateral spherical triangle is also equiangular;* and, conversely, *If the angles of a spherical triangle are equal the triangle is equilateral.*

PROPOSITION XVI.

581. ***Theorem.***—*On the same or on equal spheres two isosceles triangles having two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, can be superimposed, and are consequently equal.*

Dem.—In the triangles ABC and AB′C′, let AB = AC, AB′ = AC′; and let AB = AB′, BC = B′C′, and angle ABC = AB′C′; then can the triangle AB′C′ be superimposed upon ABC. For, since the triangles are isosceles, we have angle ABC = ACB, AB′C′ = AC′B′, and, as by hypothesis ABC = AB′C′, these four angles are equal each to each. For a like reason AB = AC = AB′ = AC′. Now, applying AC′ to its equal AB, the extremity A at A and C′ at B, with the angle B′ on the same side of AB as C, the convexities of the arcs AC′ and AB being the same, and in the same direction, the arcs will coincide. Then, as angle AC′B′ = ABC, C′B′ will take the direction BC, and since these arcs are equal by hypothesis, B′ will fall at C. Hence B′A will fall in CA, as only one arc of a great circle can pass between C and A, and the triangle AB′C′ is superimposed upon ABC; wherefore they are equal. [Let the student give the application when other parts are assumed equal.]

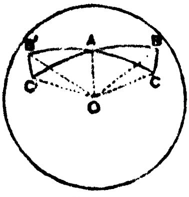


FIG. 332.

582. Symmetrical Spherical Triangles **are such as have the parts (sides and angles) of the one respectively equal to the parts of the other, but arranged in a different order, so that the triangles are not capable of superposition.**

ILL.—In *Fig.* 333, **ABC** and **A'B'C'** represent symmetrical spherical triangles. In these triangles **A** = **A'**, **B** = **B'**, **C** = **C'**, **AC** = **A'C'**, **AB** = **A'B'**, and **BC** = **B'C'**; nevertheless we cannot conceive one triangle superimposed upon the other. Thus, were we to make the attempt by placing **A'B'** in its equal **AB**, **A'** at **A**, and **B'** at **B**, the angle **C'** would fall on the opposite side of **AB** from **C**. Now, we cannot revolve **A C B** on **AB** (or its chord), and thus make the two coincide, for this would bring their convexities together. Nor can we make them coincide by reversing **A'B'C'**, and placing **B'** at **A**, and **A'** at **B**. For, although these two arcs will thus coincide, as the angle **B** is not equal to **A**, **B'C'** will not fall in **AC**; and, again, if it did, **C'** would not fall at **C**, since **B'C'** and **AC** are not equal.

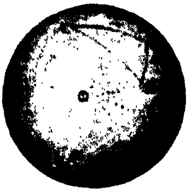


FIG. 333.

But, considering the triangles **ABC** and **A'B'C'** in *Fig.* 334, in which **A** = **A'**, **B** = **B'**, **C** = **C'**, **AC** = **A'C'**, **AB** = **A'B'**, and **BC** = **B'C'**, we can readily conceive the latter as superimposed upon the former. [The student should make the application.] Now, the two triangles are equal in each case, as will subsequently appear of the former. Such triangles as those in *Fig.* 333 are called *symmetrically equal*, while the latter are said to be equal by *superposition*.

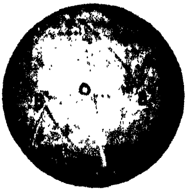


FIG. 334.

Fig. 335 represents the same triangles as *Fig.* 334, and exhibits a complete projection* of the semicircumferences of which the sides of the triangles are arcs. The student should become perfectly familiar with it, and be able to draw it readily. Thus, *a*AB'*b* is the projection of the semicircumference of which **AB** is an arc, *a*AC*c* of the semicircumference of which **AC** is an arc, etc., etc.

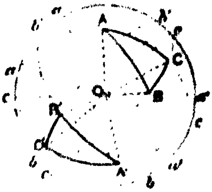


FIG. 335.

PROPOSITION XVII.

583. *Theorem.*—*Symmetrical spherical triangles are equivalent.*

---

* To understand what is meant by the projection of these lines, conceive a hemisphere with its base on the paper, and represented by the circle *abc*, and all the arcs raised up from the paper as they would be on the surface of such a hemisphere. Thus, considering the arc *a*AB*b*, the ends *a* and *b* would be in the paper just where they are, but the rest of the arc would be off the paper, as though you could take hold of **B** and raise it from the paper while *a* and *b* remain fixed. The lines in the figure are representations of lines on the surface of such a hemisphere, as they would appear to an eye situated in the axis of the circle *abc*, and at an infinite distance from it; that is, just as if each point in the lines dropped *perpendicularly* down upon the paper. Arcs of great circles perpendicular to the base are projected in straight lines passing through the centre, and oblique arcs are projected in ellipses. See *Spherical Trigonometry* (*97-100*).

Dem.—Let ABC and A'B'C' be two symmetrical spherical triangles, with AB = A'B', AC = A'C'. BC = B'C', A = A', B = B', and C = C'; then are they equivalent.

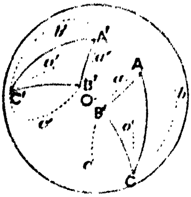


Fig. 336.

For, pass circumferences of small circles through the vertices A, B, C and A', B', C', as *abc* and *a'b'c'*, of which *o* and *o'* are the poles. [The student should execute this on the spherical blackboard.] Now, by reason of the mutual equality of the sides, the *chord* AC = *chord* A'C', *chord* AB = *chord* A'B', and *chord* BC = *chord* B'C', and as the small circles are circumscribed about the equal *plane* triangles ABC and A'B'C', these circles are equal. Hence, *o*A = *o'*A' = *o*B = *o'*B' = *o*C = *o'*C'. The triangle A*o*B is therefore equal to A'*o'*B', B*o*C = B'*o'*C', and A*o*C = A'*o'*C'. [The student should make the application of these equal triangles.] Hence, ABC is equivalent to A'B'C', as the two are composed of equal parts.

If the poles of the small circles fell without the given triangles, ABC would be equivalent to the sum of two of the partial triangles minus the third.

PROPOSITION XVIII.

***584. Theorem.**—On the same or equal spheres, two spherical triangles having two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, are equal, or symmetrical and equivalent.*

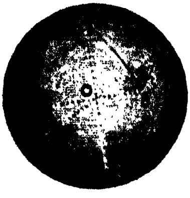


Fig. 337.

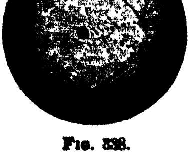


Fig. 338.

Dem.—Let ABC and A'B'C', *Fig.* 337, be two spherical triangles having AB = A'B', AC = A'C', and A = A'. In this case, as the parts are similarly arranged, by placing AC in its equal A'C', AB will fall in its equal A'B' (as A = A'), and the two triangles will coincide. Hence, they are equal. Again, let the two triangles be ABC and A'B'C', *Fig.* 338, in which AB = A'B', AC = A'C', and A = A', the parts not being similarly arranged, so that the triangles are incapable of superposition. Thus, if AB is placed in its equal A'B', A at A', and B at B', C and C' will fall on *opposite* sides of AB. We may, however, construct ABC, *Fig.* 337, symmetrical with A'B'C' in this figure, and apply ABC of *Fig.* 338 to it, and find that they coincide. Now, ABC, *Fig.* 337, and A'B'C', *Fig.* 338, are equivalent (*583*); hence ABC, *Fig.* 338, is equivalent to A'B'C', *Fig.* 338.

585. Sch.—This proposition is virtually the same as (*446*) concerning triedrals. Thus, in *Fig.* 337, drawing the radii AO, BO, CO, A'O, B'O, and C'O, two triedrals are formed, having the facial angle $AOB = A'OB'$, $AOC = A'OC'$, the included diedrals equal, and the parts similarly disposed, whence the triedrals are equal. In like manner the triedral O-ABC, *Fig.* 338, is symmetrical and equivalent to O-A'B'C', *Fig.* 338. Hence, in either case, all the parts of one spherical triangle are equal to all the parts of the other, each to each.

PROPOSITION XIX.

586. Theorem.—*On the same, or on equal spheres, two spherical triangles having two angles and the included side of the one equal to two angles and the included side of the other, each to each, are equal, or symmetrical and equivalent.*

DEM.—Using the same triangles as in the preceding proposition, the student should be able to make the application directly, when the parts are similarly disposed, and when not similarly disposed, he should be able to show that ABC, of *Fig.* 338, can be applied to ABC, *Fig.* 337, symmetrical with A'B'C', *Fig.* 338.

587. SCH.—This proposition is also virtually the same as (*447*) concerning triedrals. Let the student point out the identity.

PROPOSITION XX.

588. Theorem.—*On the same, or on equal spheres, if two spherical triangles have two sides of the one equal to two sides of the other, each to each, and the included angles unequal, the third sides are unequal, and the greater third side belongs to the triangle having the greater included angle.* Conversely, *If the two sides are equal, each to each, and the third sides unequal, the angles included by the equal sides are unequal, and the greater belongs to the triangle having the greater third side.*

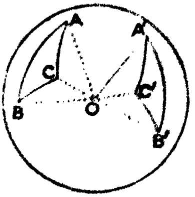


FIG. 339.

DEM.—In the triangles ABC and A'B'C', let $AB = A'B'$, $AC = A'C'$, and $A > A'$; then is $BC > B'C'$. For, join the vertices with the centre, forming the two triedrals O-ABC and O-A'B'C'. In these triedrals $AOB = A'OB'$, $AOC = A'OC'$, being measured by equal arcs; and $C\text{-}AO\text{-}B > C'\text{-}A'O\text{-}B'$, having the same measures as A and A' (*558*). Hence $COB > C'OB'$ (*449*). Therefore CB, the measure of COB, is greater than C'B', the measure of C'OB'.

In like manner, the same sides of the triangles, and consequently the same facial angles of the triedrals, being granted equal, and

$BC > B'C'$, $A > A'$. For, BC being greater than B'C', $COB > C'OB'$; whence $B\text{-}AO\text{-}C > B'\text{-}A'O\text{-}C'$ (*450*), or A is greater than A'.

PROPOSITION XXI.

589. Theorem.*—On the same, or on equal spheres, two spherical triangles having the sides of the one respectively equal to the sides of the other, or the angles of the one respectively equal to the angles of the other, are equal, or symmetrical and equivalent.*

DEM.—The sides of the triangles being equal, the facial angles of the triedrals at the centre are equal, whence the triedrals are equal or symmetrical (*451*). Consequently the angles of the triangles are equal, and the triangles are equal, or symmetrical and equivalent.

Again, the triangles being mutually equiangular, the triedrals have their diedrals mutually equal; whence the triedrals are equal or symmetrical (*452*). Therefore, the sides of the triangles are mutually equal, and the triangles are equal, or symmetrical and equivalent. (See *Figs.* 333, 334.)

PROPOSITION XXII.

590. Theorem.*—On spheres of different radii, mutually equiangular triangles are similar* (not equal).

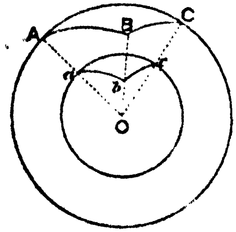


FIG. 340.

DEM.—Let O be the common centre of two unequal spheres; and let ABC be a spherical triangle on the surface of the outer. Draw the radii AO, BO, and CO, constructing the triedral O-ABC. Now, the intersections of these faces with the surface of the inner sphere will constitute a triangle which is mutually equiangular with ABC. Thus, $A = a$, $B = b$, and $C = c$, since in each case the corresponding diedrals are the same. From the similar sectors aOb, AOB, we have $ab : AB :: aO : AO$; and, in like manner, $ac : AC :: aO : AO$. Whence, $ab : AB :: ac : AC$. So, also, $ab : AB :: bO : BO$, and $bc : BC :: bO : BO$; whence, $ab : AB :: bc : BC$. Thus we see that ABC and abc, having their angles equal each to each, have also their sides proportional: therefore they are similar.

POLAR OR SUPPLEMENTAL TRIANGLES.

591. One triangle is polar to another when the vertices of one are the poles of the sides of the other. Such triangles are also called *supplemental*, since the angles of one are the supplements of the sides opposite in the other, as will appear hereafter.

PROPOSITION XXIII.

592. ***Problem.****—Having a spherical triangle given, to draw its polar.*

SOLUTION.—Let **ABC** be the given triangle.* From **A** as a pole, with a quadrant strike an arc, as **C′B′**. From **B** as a pole, with a quadrant strike the arc **C′A′**; and from **C**, the arc **A′B′**. Then is **A′B′C′** polar to **ABC**.

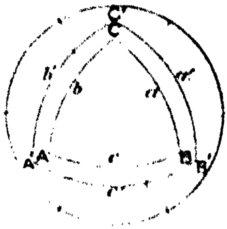


FIG. 341

593. COR.—*If one triangle is polar to another*, conversely, *the latter is polar to the former*; i. e., *the relation is reciprocal.*

Thus, **A′B′C′** being polar to **ABC**; reciprocally, **ABC** is polar to **A′B′C′**; that is, **A′** is the pole of **CB**, **B′** of **AC**, and **C′** of **AB**. For every point in **A′B′** is at a quadrant's distance from **C**, and every point in **A′C′** is at a quadrant's distance from **B**. Hence, **A′** is at a quadrant's distance from the two points **C** and **B** of **CB**, and is therefore its pole. [In like manner the student should show that **B′** is the pole of **AC**, and **C′** of **AB**.]

594. SCH.—By producing each of the arcs struck from the vertices of the given triangles sufficiently, *four* new triangles will be formed, viz., **A′B′C′**, **QC′B′**, **PC′A′**, and **RA′B′**. Only the first of these is called polar to the given triangle. It is easy to observe the relation of any of the parts of any one of the other three triangles to the parts of the polar. Thus, **QC′** = 180° − b', **QB′** = 180° − c', **QC′B′** = 180° − **B′C′A′**, **QB′C′** = 180° − **C′B′A′**, and **Q** = **A′** = 180° − a, as will appear hereafter.

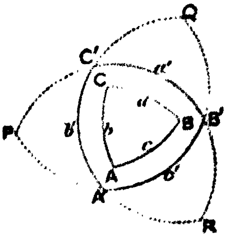


FIG. 342.

---

* This should be executed on a sphere. Few students get clear ideas of polar triangles without it. Care should be taken to construct a variety of triangles as the given triangle, since the polar triangle does not always lie in the position indicated in the figure here given. Let the given triangle have one side considerably greater than 90°, another somewhat less, and the third quite small. Also, let each of the sides of the given triangle be greater than 90°.

PROPOSITION XXIV.

595. *Theorem.*—*Any* ANGLE *of a spherical triangle is the supplement of the* SIDE *opposite in its polar triangle; and any* SIDE *is the supplement of the* ANGLE *opposite in the polar triangle.*

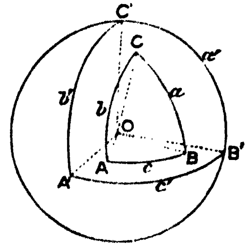


FIG. 343.

DEM.—Let ABC and A'B'C' be two spherical triangles polar to each other; and let the sides of each be designated as a, b, c, a', b', c', a being opposite A, a' opposite A', b opposite B, etc. Then $A = 180° - a'$, $B = 180° - b'$, $C = 180° - c'$, $a = 180° - A'$, $b = 180° - B'$, and $c = 180° - C'$.

For, join the vertices of the triangles with the centre of the sphere, thus forming the triedrals O-ABC, and O-A'B'C'. These triedrals are supplemental; for, A being the pole of C'B', AO is the axis of the great circle of which C'B' is an arc (?), hence is perpendicular to the plane C'OB', and consequently to OB' and OC' (?). In like manner, BO is perpendicular to the plane A'OC', and hence to OA' and OC'. So, also, CO is perpendicular to OA' and OB'. Now, these triedrals being supplementary, the diedral B-AO-C is the supplement of the facial angle C'OB' (*438*); or, since the diedral B-AO-C is the same as the spherical angle A, and the facial angle C'OB' is measured by a', A is the supplement of a', *i. e.*, $A = 180° - a'$. For like reasons, $B = 180° - b'$, and $C = 180° - c'$. [Let the student give them in full.] Again, the diedral B'-A'O-C' is the supplement of the facial angle COB (*438*); whence $A' = 180° - a$. In like manner $B' = 180° - b$, and $C' = 180° - c$.

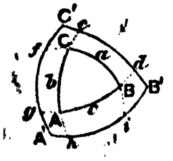


FIG. 344.

SECOND DEMONSTRATION.—Let ABC and A'B'C' be two polar triangles. Let CB, CA, and AB be represented by a, b, and c respectively, and C'B', C'A', and A'B' by a', b', and c'. To show that $A = 180° - a'$, produce b and c, if necessary, till they meet the side a', of the triangle polar to ABC, in e and d. Now A is measured by ed (*560*). But, since $C'd = 90°$, and $B'e = 90°$, $C'd + B'e$, or $C'B' + ed = 180°$; whence transposing, and putting a' for C'B', we have $ed = A = 180° - a'$. In like manner $C'g + A'f = C'A' + fg = 180°$; whence $fg = B = 180° - C'A'$, or $180° - b'$. So, also, $C = 180° - c'$. To show that $A' = 180° - a$, consider that A' being the pole of CB, fi is the measure of A'. Now $Bf = 90°$ (?), and $Ci = 90°$; whence $Bf + Ci = 180°$. But $Bf + Ci = fi + a$, wherefore $fi + a = 180°$, and transposing, and putting A' for fi, we have $A' = 180° - a$. In like manner we may show that $B' = 180° - b$, and $C' = 180° - c$. [The student should give the details.]

QUADRATURE OF THE SURFACE OF THE SPHERE.

596. The *Quadrature** of a surface is the same as finding its area. The term is applied under the conception that the process consists in finding a *square* which is equivalent to the given surface.

PROPOSITION XXV.

597. *Lemma.*—*The surface generated by the revolution of a regular semi-polygon of an even number of sides, about the diameter of the circumscribed circle as an axis, is equivalent to the circumference of the inscribed circle multiplied by the axis.*

DEM.—Let ABCDE be one half of a regular octagon, AE being the diameter of the circumscribing circle. If the semi-perimeter ABCDE be revolved about AE as an axis, the surface generated will be $2\pi r \times AE$, r being the radius of the inscribed circle, as aO, or bO.

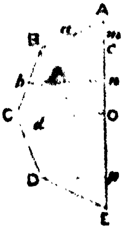


FIG. 345.

This surface is composed of the convex surfaces of cones and frustums of cones. Thus AB generates the surface of a cone, BC the frustum of a cone, etc. Let a and b be the middle points of AB and BC, and draw am, Bc, bn, and CO perpendicular to the axis, and Bd parallel to it. Also draw the radii of the inscribed circle, aO and bO. Indicate the surfaces generated by the sides, as *Surf.* AB, *Surf.* BC, etc. The areas of these surfaces are:

$$Surf.\ AB = 2\pi \times am \times AB\ (516), \qquad (1)$$

$$Surf.\ BC = 2\pi \times bn \times BC\ (518), \text{ etc.} \qquad (2)$$

Now, from the similar triangles Oam and BAc,

We have $aO : AB :: am : Ac$, or $2\pi \times aO : AB :: 2\pi \times am : Ac$;

Whence $2\pi \times am \times AB = 2\pi r \times Ac$, putting r for aO.

Also, from the similar triangles Obn and CBd,

We have $bO : BC :: bn : Bd\ (= cO)$, or $2\pi \times bO : BC :: 2\pi \times bn : cO$;

Whence $2\pi \times bn \times BC = 2\pi r \times cO$, putting r for bO.

Substituting these values in (1) and (2), we obtain

$$Surf.\ AB = 2\pi r \times Ac,$$

$$Surf.\ BC = 2\pi r \times cO,$$

And, in like manner, $Surf.\ CD = 2\pi r \times Op$,

And, $Surf.\ DE = 2\pi r \times pE$.

Adding, $Surf.\ ABCDE = 2\pi r\,(Ac + cO + Op + pE) = 2\pi r \times AE.$

Finally, since the same course of reasoning is applicable to the semi-polygons of 16, 32, 64, etc., sides, the truth of the proposition is established.

---

* Latin *quadratus*, squared.

598. Sch.—This proposition is only a particular case of surfaces generated by *any* broken line revolving about an axis; and the general proposition can be established in a manner altogether similar to the method given above. But this case is all that we need for our present purpose.

PROPOSITION XXVI.

599. Theorem.—*The surface of a sphere is equivalent to four great circles; that is, to* $4\pi R^2$, R *being the radius of the sphere.*

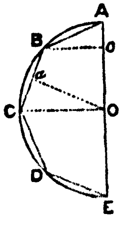


Fig. 346.

Dem.—Let the semicircumference ABCDE revolve upon the diameter AE, and thus generate the surface of a sphere. Conceive the half of a regular octagon inscribed in the semicircle. Call the radius of the inscribed circle, as aO, r, and let AO = R. The surface generated by the broken line ABCDE is, by the last proposition, $2\pi r \times 2R = 4\pi r R$. Now, conceive the arcs AB, BC, etc., bisected, and the chords drawn, and let r' be the radius of the circle inscribed in the regular polygon thus formed. The surface generated by this semi-polygon will be $4\pi r'R$. By repeating the bisections, the broken line approximates to the semicircumference, the radius of the inscribed circle to R, and the surface generated to the surface of the sphere, the three quantities reaching their limits at the same time. Hence *at* the limit we have

Surf. of sphere $= 2\pi R \times 2R = 4\pi R^2$. Q. E. D.

600. Cor. 1.—*The area of the surface of a sphere is equivalent to the circumference of a great circle multiplied by the diameter, that is,* $2\pi R \times 2R$, *as above.*

601. Cor. 2.—*The surfaces of spheres are to each other as the squares of their radii.*

Thus, if R and R′ are the radii of two spheres, the surfaces are $4\pi R^2$ and $4\pi R'^2$. Now, $4\pi R^2 : 4\pi R'^2 :: R^2 : R'^2$.

---

602. Def.—***A Zone*** is the portion of the surface of a sphere included between the circumferences of two parallel circles of a sphere. The *altitude* of a zone is the distance between the parallel circles forming its bases.

Ill.—The surface generated by arc CB, or any arc of the circle ABCDE, *Fig.* 346, in its revolution, conforms to the definition, and is a zone. Such a portion of the surface as is generated by AB is called a zone with one base, the circle whose circumference would form the upper base having become tangent to the sphere.

PROPOSITION XXVII.

603. Theorem.—*The area of a zone is to the area of the surface of the sphere as the altitude of the zone is to the diameter of the sphere; which gives for the area of a zone* $2\pi aR$, *a being the altitude of the zone, and* R *the radius of the sphere.*

DEM.—It is evident that in passing to the limit the surface generated by such a portion of the broken line as would lie between C and B, *Fig.* 346, would be measured by the circumference of the inscribed circle multiplied by cO Hence, at the limit, the zone generated by arc BC is measured by $2\pi R \times cO$, that is, it is such a part of the surface of the sphere as cO is of AE, or $2R$. Letting a represent the altitude cO, the fraction $\frac{a}{2R}$ represents the part of the surface of the sphere constituting the area of the zone. Hence, $4\pi R^2 \times \frac{a}{2R}$, which equals $2\pi aR$, is the area of the zone.

604. COR.—*On the same or on equal spheres, zones are to each other as their altitudes.*

OF LUNES.

605. A Lune is a portion of the surface of a sphere included by two semicircumferences of great circles.

The surface AmBn is a lune.

606. The *Angle of the Lune* is the angle included by the arcs which form its sides; or, what is the same thing, the measure of the diedral included between the great circles.

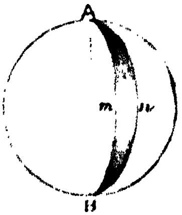


FIG. 347.

Thus, the spherical angle mAn, or the measure of the diedral m-AB-n, is the angle of the lune AmBn.

PROPOSITION XXVIII.

607. Theorem.—*The area of a lune is to the area of the surface of the sphere on which it is situated, as the angle of the lune is to four right angles.*

DEM.—Let ACEB be a lune whose angle is the spherical angle CAB, or what

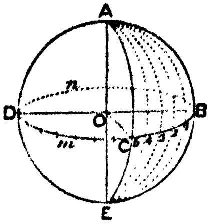


FIG. 348.

is the same thing, the plane angle BOC measured by the arc CB, of which A is the pole; then is

lune ACEB : *surface of sphere* :: CAB : 4 *right angles*.

For, suppose the arc CB commensurable with the circumference BCmDn, and suppose that they are to each other as 5 : 24. Dividing BC into 5 equal arcs, and the entire circumference BCmDn into 24 arcs of the same length, and passing arcs of great circles through A and these points of division, the lune will be divided into 5 equal lunes, and the entire surface into 24 equal lunes of the same size. That these lunes are equal to each other is evident from the fact that they are composed of equal isosceles triangles. Hence,

lune ACEB : *surface of sphere* : : 5 : 24.

Now, *angle* BOC : 4 *right angles* :: BC (= 5) : BCmDn (= 24).

Therefore, *lune* ACEB : *surface of sphere* :: BOC (or CAB) : 4 *right angles*,

since the circumference measures 4 right angles.

If BC has no finite common measure with the circumference, we may divide it into any number of equal arcs, bisect these arcs, then bisect the last formed, and continue the process of bisection (in conception) to any required extent; and as, when any one of the arcs thus obtained is applied to the circumference, if it is not an exact measure, the remainder is less than the arc, we can continue the subdivision of BC (in conception) until this remainder is less than any assignable quantity. Hence, we may always consider the arc BC as commensurable with the circumference by making the measure infinitesimal.

608. COR.—*The sum of several lunes on the same sphere is equal to a lune whose angle is the sum of the angles of the lunes; and the difference of two lunes is a lune whose angle is the difference of their angles.*

609. SCH. 1.—The case in which the arc measuring the angle of the lune is incommensurable with the circumference, may be treated as in (***206***), by the method of reasoning called the *Reductio ad absurdum, i. e.*, by showing a thing to be true, since it would be absurd to suppose it untrue.

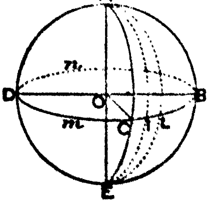


FIG. 349.

Thus, there is some arc to which the circumference bears the same ratio as the surface of the sphere does to the surface of the lune. If that arc be not BC let it be BL, an arc *less* than BC, so that

surface of sphere : *lune* ACEB : : BCmDn : BL. (1)

Conceive the circumference BCmDn divided into equal parts, each of which is less than CL, the assumed difference between BC and BL. Then conceive one of these equal parts applied to BC as a measure, beginning at B. Since the measure is less than LC, one point of division, at least, will fall between L and C. Let I be such a point, and pass the arc of a great circle through A and I.

Now, *surface of sphere* : *lune* AIEB : : BCmDn : BI, (2)

since the arc BI is commensurable with the circumference. In (1) and (2), the antecedents being equal, the consequents should be proportional, hence we should have

lune ACEB : *lune* AIEB : : BL : BI.

But this is absurd, since *lune* ACEB > lune AIEB, whereas BL < BI. In a similar manner we can show that

surface of sphere is not to *lune* ACEB : : BCmDn : any arc greater than BC.

Hence, as the fourth term can neither be less nor greater than BC, it must be equal to BC, and we have

surface of sphere : *lune* ACEB : : BCmDn : BC,

i. e., as 4 right angles, to the angle of the lune.

610. Sch. 2.—*To obtain the area of a lune whose angle is known, on a given sphere*, find the area of the sphere, and multiply it by the ratio of the angle of the lune (in degrees) to 360°. Thus, R being the radius of the sphere, $4\pi R^2$ is the surface of the sphere; and the lune whose angle is 30° is $\frac{30}{360}$ or $\frac{1}{12}$ the surface of the sphere, *i. e.*, $\frac{1}{12}$ of $4\pi R^2 = \frac{1}{3}\pi R^2$.

PROPOSITION XXIX.

611. Theorem.—*If two semicircumferences of great circles intersect on the surface of a hemisphere, the sum of the two opposite triangles thus formed is equivalent to a lune whose angle is that included by the semicircumferences.*

Dem.—Let the semicircumferences CEB and DEA intersect at E on the surface of the hemisphere whose base is CABD; then the sum of the triangles CED and AEB is equivalent to a lune whose angle is AEB.

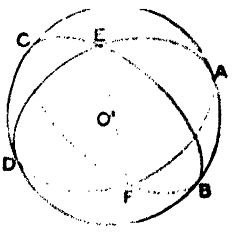


Fig. 200.

For, let the semicircumferences CEB and DEA be produced around the sphere, intersecting on the opposite hemisphere, at the extremity F of the diameter through E. Now, FBEA is a lune whose angle is AEB. Moreover, the triangle AFB is equivalent to the triangle DEC: since angle AFB = AEB = DEC, side AF = side ED, each being the supplement of AE; and BF = CE, each being the supplement of EB. Hence, the sum of the triangles CED and AEB is equivalent to the lune FBEA. Q. E. D.

PROPOSITION XXX.

612. Theorem.—*The area of a spherical triangle is to the area of the surface of the hemisphere in which it is situated, as its spherical excess is to four right angles, or* 360°.

DEM.—Let ABC be a spherical triangle whose angles are represented by A, B, and C; then is

area ABC : *surf. of hemisphere* : : A + B + C − 180° : 4 *right angles*, or 360°.

Let *lune* A represent the lune whose angle is the angle A of the triangle, *i. e.*, angle CAB, and in like manner understand *lune* B and *lune* C.

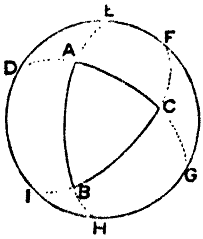


FIG. 351.

Now, triangle AHC + AED = *lune* A (*611*),

BHI + BEF = *lune* B,

CGF + CDI = *lune* C.

Adding, 2ABC + *hemisphere* = *lune* (A + B + C)*, (1) since the six triangles AHC, AED, BHI, BEF, CGF, and CDI, make the whole hemisphere and 2ABC besides, ABC being reckoned *three* times. From (1), we have by transposing and remembering that a hemisphere is a lune whose angle is 180°, and dividing by 2,

$$\text{ABC} = \tfrac{1}{2}\,lune\,(\text{A} + \text{B} + \text{C} - 180^\circ).$$

But, by (*607*),

½ *lune* (A + B + C − 180°) : *surf. of hemisph.* : : A + B + C − 180° : 4 *right angles.*

Therefore, ABC : *surf. of hemisph.* : : A + B + C − 180° : 4 *right angles.*

613. SCH. 1.—*To find the area of a spherical triangle* on a given sphere, the angles of the triangle being given, we have simply to multiply the area of the hemisphere, *i. e.*, $2\pi R^2$, by the ratio of the spherical excess to 360°. Thus, if the angles are A = 110°, B = 80°, and C = 50°, we have

$$area\ \text{ABC} = 2\pi R^2 \times \frac{\text{A} + \text{B} + \text{C} - 180^\circ}{360^\circ} = 2\pi R^2 \times \frac{60}{360} = \tfrac{1}{3}\pi R^2.$$

614. SCH. 2.—This proposition is usually stated thus: *The area of a spherical triangle is equal to its spherical excess multiplied by the trirectangular triangle.* When so stated the spherical excess is to be estimated in terms of the right angle; *i. e.*, having subtracted 180° from the sum of its angles, we are to divide the remainder by 90°, thus getting the spherical excess in right angles. In the example in the preceding scholium, the spherical excess estimated in this way would be $\frac{110^\circ + 80^\circ + 50^\circ - 180^\circ}{90^\circ} = \frac{2}{3}$; and the area of the triangle would be $\frac{2}{3}$ of the trirectangular triangle. Now, the trirectangular triangle being $\frac{1}{8}$ of the surface of the sphere (577) is $\frac{1}{8}$ of $4\pi R^2$, or $\frac{1}{2}\pi R^2$. This multiplied by $\frac{2}{3}$ gives $\frac{1}{3}\pi R^2$, the same as above.

---

* This signifies the lune whose angle is A + B + C, which is of course the sum of the three lunes whose angles are A, B, and C.

The proportion,

$$ABC : \textit{surf. of hemisph.} :: A + B + C - 180^\circ : 360^\circ,$$

is readily put into a form which agrees with the enunciation as given in this scholium. Thus, *surf. of hemisph.* $= 2\pi R^2$, whence

$$ABC = 2\pi R^2 \times \frac{A + B + C - 180^\circ}{360^\circ} = \frac{1}{2}\pi R^2 \times \frac{A + B + C - 180^\circ}{90^\circ}.$$

VOLUME OF SPHERE.

PROPOSITION XXXI.

615. Theorem.—*The volume of a sphere is equal to the area of its surface multiplied by $\frac{1}{3}$ of the radius, that is, $\frac{4}{3}\pi R^3$, R being the radius.*

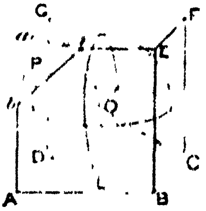


FIG. [illegible]

DEM.—Let OL = R be the radius of a sphere. Conceive a circumscribed cube, that is, a cube whose faces are tangent planes to the sphere. Draw lines from the vertices of each of the polyedral angles of the cube, to the centre of the sphere, as BO, CO, DO, AO, etc. These lines are the edges of six pyramids, having for their bases the faces of the cube, and for a common altitude the radius of the sphere ([illegible]). Hence the volume of the circumscribed cube is equal to its surface multiplied by $\frac{1}{3}$R.

Again, conceive each of the polyedral angles of the cube truncated by planes *tangent to the sphere*. A new circumscribed solid will thus be formed, whose volume will be nearer that of the sphere than is that of the circumscribed cube. Let *abc* represent one of these tangent planes. Draw from the polyedral angles of this new solid, lines to the centre of the sphere, as *a*O, *b*O, and *c*O, etc.; these lines will form the edges of a set of pyramids whose bases constitute the surface of the solid, and whose common altitude is the radius of the sphere ([illegible]). Hence the volume of this solid is equal to the product of its surface (the sum of the bases of the pyramids) into $\frac{1}{3}$R.

Now, this process of truncating the angles by tangent planes may be conceived as continued indefinitely; and, to whatever extent it is carried, it will *always* be true that the volume of the solid is equal to its surface multiplied by $\frac{1}{3}$R. Therefore, as the sphere is the limit of this circumscribed solid, we have the volume of the sphere equal to the surface of the sphere, which is $4\pi R^2$, multiplied by $\frac{1}{3}$R, *i. e.*, to $\frac{4}{3}\pi R^3$. Q. E. D.

616. COR.—*The surface of the sphere may be conceived as consisting of an infinite number of infinitely small plane faces, and the volume as composed of an infinite number of pyramids having these faces for their bases, and their vertices at the centre of the sphere, the common altitude of the pyramids being the radius of the sphere.*

617. A Spherical Sector is a portion of a sphere generated by the revolution of a circular sector about the diameter around which the semicircle which generates the sphere is conceived to revolve. It has a zone for its base; and it may have as its other surfaces one, or two, conical surfaces, or one conical and one plane surface.

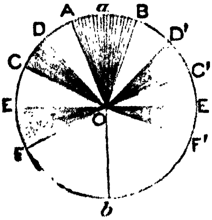


FIG. 853.

ILL.—Thus let ab be the diameter around which the semicircle aCb revolves to generate the sphere. The solid generated by the circular sector AOa will be a spherical sector having a zone (AB) for its base; and for its other surface, the conical surface generated by AO. The spherical sector generated by COD, has the zone generated by CD for its base; and for its other surfaces, the concave conical surface generated by DO, and the convex conical surface generated by CO. The spherical sector generated by EOF, has the zone generated by EF for its base, the plane generated by EO for one surface, and the concave conical surface generated by FO for the other.

618. A Spherical Segment is a portion of the sphere included by two parallel planes, it being understood that one of the planes may become a tangent plane. In the latter case, the segment has but one base; in other cases, it has two. A spherical segment is bounded by a zone and one, or two, plane surfaces.

PROPOSITION XXXII.

619. Theorem.—*The volume of a spherical sector is equal to the product of the zone which forms its base into one-third the radius of the sphere.*

DEM.—A spherical sector, like the sphere itself, may be conceived as consisting of an infinite number of pyramids whose bases make up its surface, and whose common altitude is the radius of the sphere. Hence, the volume of the sector is equal to the sum of the bases of these pyramids, that is, the spherical surface of the sector, multiplied by one-third their common altitude, which is one-third the radius of the sphere. Q. E. D.

620. Cor.—*The volumes of spherical sectors of the same or equal spheres are to each other as the zones which form their bases; and, since these zones are to each other as their altitudes* (**604**), *the sectors are to each other as the altitudes of the zones which form their bases.*

PROPOSITION XXXIII.

Theorem.—*The volume of a spherical segment of one is* $\pi A^2(R - \frac{1}{3}A)$, *A being the altitude of the segment, and* R *the ra- of the sphere.*

Dem.—Let **CO** = R, and **CD** = A; then is the volume of the spherical ment generated by the revolution of **CAD** about **CO** equal to $\pi A^2(R - \frac{1}{3}A)$.

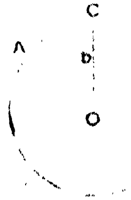


Fig. 324.

For, the volume of the spherical sector generated by **AOC** is the zone generated by **AC**, multiplied by $\frac{1}{3}R$, or $2\pi AR \times \frac{1}{3}R = \frac{2}{3}\pi AR^2$. From this we must subtract the cone, the radius of whose base is **AD**, and whose altitude is **DO**. To obtain this, we have **DO** $= R - A$: whence, from the right angled triangle **ADO**, **AD** $= \sqrt{R^2 - (R - A)^2} = \sqrt{2AR - A^2}$. Now, the volume of this cone is

$$\times\ \pi \overline{AD}^2, \text{ or } \tfrac{1}{3}\pi(R - A)(2AR - A^2) = \tfrac{1}{3}\pi(2AR^2 - 3A^2R + A^3).$$

Subtracting this from the volume of the spherical sector, we have

$$- \tfrac{1}{3}\pi(2AR^2 - 3A^2R + A^3) =$$
$$\pi(A^2R - \tfrac{1}{3}A^3) = \pi A^2(R - \tfrac{1}{3}A). \quad \text{Q. E. D.}$$

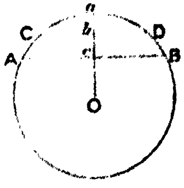


Fig. 325.

Sch.—The volume of a spherical segment with two bases is readily obtained by taking the difference between two segments of one base each. Thus, to obtain the volumes of the segment generated by the revolution of *b***CA***c* about *a***O**, take the difference of the segments whose altitudes are *ac* and *ab*.

1. What is the circumference of a small circle of a sphere whose diameter is 10, the circle being at 3 from the centre?

Ans., 25.1328.

2. Construct on the spherical blackboard a spherical angle of 60°. Of 90°. Of 120°. Of 250°.

SUG'S.—Let P be the point where the vertex of the required angle is to be situated. With a quadrant strike an arc from P, which shall represent one side of the required angle. From P as a pole, with a quadrant, strike an arc from the side before drawn, which shall measure the required angle. On this last arc lay off from the first side the measure of the required angle,* as 60°, 45°, etc. Through the extremity of this arc and P pass a great circle (*548*). [The student should not fail to give the reasons, as well as *do* the work.]

3. On the spherical blackboard construct a spherical triangle ABC, having AB = 100°, AC = 80°, and A = 58°.

4. Construct as above a spherical triangle ABC, having AB = 75°, A = 110°, and B = 87°.

5. Construct as above, having AB = 150°, BC = 80°, and AC = 100°. Also having AB = 160°, AC = 50°, and BC = 85°.

6. Construct as above, having A = 52°, AC = 47°, and CB = 40°.

SUG'S.—Construct the angle A as before taught, and lay off AC from A equal to 47°, with the tape. This determines the vertex C. From C, as a pole, with an arc of 40°, describe an arc of a small circle; in this case this arc will cut the opposite side of the angle A in two places. Call these points B and B′. Pass circumferences of great circles through C, and B, and B′. There are two triangles, ACB and ACB′.

NOTE.—The teacher can multiply examples like the three preceding at pleasure. This exercise should be continued till the pupil can draw a spherical triangle as readily as a plane triangle.

7. What is the area of a spherical triangle on the surface of a sphere whose radius is 10, the angles of the triangle being 85°, 120°, and 150°? *Ans.*, 305.4 +.

8. What is the area of a spherical triangle on a sphere whose diameter is 12, the angles of the triangle being 82°, 98°, and 100°?

9. A sphere is cut by 5 parallel planes at 7 from each other. What are the relative areas of the zones? What of the segments?

10. Considering the earth as a sphere, its radius would be 3958 miles, and the altitudes of the zones, North torrid = 1578, North temperate = 2052, and North frigid = 328 miles. What are the relative areas of the several zones?

SUG.—The student should be careful to discriminate between the *width* of a zone, and its altitude. The altitudes are found from their widths, as usually given in degrees, by means of trigonometry.

---

* For this purpose a tape equal in length to a semicircumference of a great circle of the sphere used, and marked off into 180 equal parts, will be convenient. A strip of paper may

11. The earth being regarded as a sphere whose radius is 3958 miles, what is the area of a spherical triangle on its surface, the angles being 120°, 130°, and 150°? What is the area of a trirectangular triangle on the earth's surface?

12. Construct on the spherical blackboard a spherical triangle ABC, having A = 59°, AC = 120°, and AB = 88°. Then construct the triangle polar to ABC.

13. Construct triangles polar to each of those in Examples 3, 4, and 5.

14. In the spherical triangle ABC given A = 58°, B = 67°, and AC = 81°; what can you affirm of the polar triangle?

15. What is the volume of a globe which is 2 feet in diameter? What of a segment of the same globe included by two parallel planes, one at 3 and the other at 9 inches from the centre?

16. Compare the convex surfaces of a sphere and its circumscribed cylinder and cone, the vertical angle of the cone being 60°.

17. Compare the volumes of a sphere and its circumscribed cube, cylinder, and cone, the vertical angle of the cone being 60°.

18. If a and b represent the distances from the centre of a sphere whose radius is r, to the bases of a spherical segment, show that the volume of the segment is $\pi[r^2(b-a) - \frac{1}{3}(b^3 - a^3)]$. See **(621,**

PART III.

AN ADVANCED COURSE IN GEOMETRY.

CHAPTER I.

EXERCISES IN GEOMETRICAL INVENTION.

SECTION I.

THEOREMS IN SPECIAL OR ELEMENTARY GEOMETRY.

623. This chapter will afford a review of Parts I. and II., while it will greatly extend the student's knowledge of geometrical facts. Great pains should be taken to secure good habits as to neatness of execution in the construction of figures, orderly and proper arrangement of thought, and in style of expression. The practice of constructing every figure upon geometrical principles—guessing at nothing—cannot be too strongly commended. As to the *form* of a geometrical argument, observe the following order:

1st. The enunciation of the theorem or problem in general terms.

2d. The elucidation of the general statement, by reference to the particular figure which it is proposed to use.

3d. A description of the figure, with reference to any auxiliary construction which is used in the demonstration or solution.

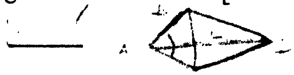
4th. The demonstration proper.

624. **If two adjacent sides of a quadrilateral are equal each to each, and the other two adjacent sides equal each to each, the diagonals intersect at right angles.**

SUG'S.—1st. Draw a quadrilateral having such sides as the data require, and draw its diagonals. 2d. State the proposition with reference to the figure.

3d. [In this case the regular third step is not required, as no auxiliary lines are necessary.] 4th. Prove that the diagonals are at right angles to each other. The demonstration is based upon a corollary in Section I., Part II., Chapter I.

625. Cor.—One of the diagonals is *bisected.* [State which one, and show why.]



626. If a parallelogram has one oblique angle, all its angles are oblique; and if it has one right angle, all its angles are right angles.

Sug's.—Let the student be careful to follow the order as heretofore given. No auxiliary construction is needed. The demonstration is based upon the doctrine of parallels.

---

627. The sum of three straight lines drawn from any point within a triangle to the vertices is less than the sum, and greater than the half sum of the three sides of the triangle.

Sug's.—The first statement is proved from (*276*) and the second from (*274*.)

---

628. A line drawn from any angle of a triangle to the middle of the opposite side, is less than the half sum of the adjacent sides, and greater than the difference between this half sum and half the third side.

Sug's.—1st. Draw a triangle, as **ABC**, bisect one side, as **AC**, and draw **BD**. 2d. Make the statement with reference to the figure. 3d. Produce **BD** until **DE** = **BD**, and draw **AE** and **EC**. 4th. The first step in the proof is to show the triangle **ADE** equal to **CBD**, and **ADB** equal to **DCE**; whence **AE** = **BC**, and **EC** = **AB**.

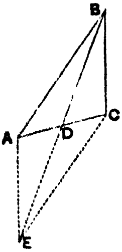


Fig. 356.

---

629. If lines be drawn from the extremities of either of the non-parallel sides of a trapezoid to the middle of the opposite side, the triangle thus formed is half the trapezoid.

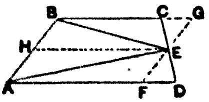


Fig. 357.

Sug's.—The third step, or construction, consists in drawing **HE** parallel to **AD** and hence to **BC** (?), and **FG** through **E** parallel to **AB**.

---

630. Any line drawn through the centre of the diagonal of a parallelogram bisects the figure.

631. Prove that the sum of the angles of a triangle is two right angles, by producing two of the sides about an angle and through this angle drawing a line parallel to the third side.

Prove the same by producing one side of the triangle and drawing a line through the exterior angle parallel to the non-adjacent side.

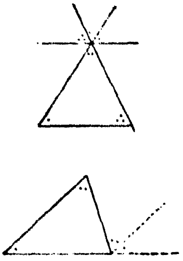


Fig. 358.

---

632. If any point, not the centre, be taken in a diameter of a circle, of all the chords which can pass through that point, that one is the least which is at right angles to the diameter.

---

633. If from any point there extend two lines tangent to a circumference, the angle contained by the tangents is double the angle contained by the line joining the points of tangency and the radius extending to one of them.

---

634. The angle included by two lines drawn from any angle of a triangle, the one bisecting the angle and the other perpendicular to the opposite side, is half the difference of the other two angles of the triangle.

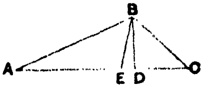


Fig. 359.

Sug's. ABD = 90° − A, whence ABD − EBD = 90° − A − EBD. Also, DBC = 90° − C, whence EBC = 90° − C + EBD = 90° − A − EBD, etc.

---

635. If three lines be drawn from the acute angles of a right angled triangle—two bisecting these angles, and a third a perpendicular to one of the bisecting lines—the triangle included by these lines will be isosceles.

Sug's.—It is to be proved that OD = CD. COD = OAC + ACO = 45°, etc.

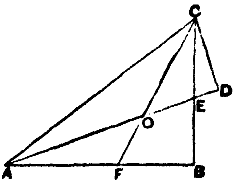


Fig. 360.

---

636. If one circumference be described on the radius of another as a diameter, any straight line extending from their point of contact to the outer circumference is bisected by the inner.

Sug.—The demonstration is based upon (*159*, *211*).

637. Prove that the sum of the angles of a regular five point star (Fig. 101) is two right angles. Show, also, that the figure formed by the intercepted portions of the lines is a regular pentagon.

638. If the sides of a regular hexagon are produced till they meet, show that the exterior figures will be equilateral triangles.

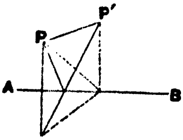


FIG. 361.

639. If from two given points on the same side of a given line, two lines be drawn meeting in the line, their sum is least when they make equal angles with the line.

640. If from two given points without a circumference, two lines be drawn meeting in the circumference, their sum is least when they make equal angles with a tangent at the common point, the points being on the opposite side of the tangent from the circle.

641. The side of an equilateral triangle inscribed in a circle is equal to the diagonal of a rhombus, whose other diagonal and each of whose sides are equal to the radius.

642. If two circumferences intersect each other, and from either point of intersection a diameter be drawn in each, the other extremities of these diameters and the other point of intersection are in the same straight line.

643. If any straight line joining two parallels be bisected, any other line through the point of bisection and included by the parallels, is bisected at the same point.

644. If the sides of any quadrilateral are bisected, the quadrilateral formed by joining the adjacent points of bisection is a parallelogram.

SUG'S.—1st. Draw a quadrilateral, bisect its sides, and join the adjacent points of bisection. 2d. State the proposition, with reference to the figure. 3d. Draw the diagonals. 4th. Give the proof. It is based on the similarity of triangles.

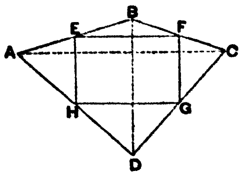


FIG. 362.

645. COR. 1.—The parallelogram is one-half the trapezium. Prove it. What figure is formed by joining the centres of EF, FC, and FC, HC, etc. ?

646. COR. 2.—Lines joining the middle points of the opposite sides of any trapezium bisect each other (?).

---

647. If two straight lines join the alternate ends of two parallels, the line joining their centres is half the difference of the parallels.

SUG'S.—We are to prove that $EF = \frac{1}{2}(CD - AB)$. $CH = EF = \frac{1}{2}(CD - AB)$.

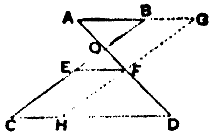


FIG. 363.

---

648. In any right-angled triangle the line drawn from the right angle to the middle of the hypotenuse is equal to one-half the hypotenuse.

---

649. The perpendiculars which bisect the three sides of a triangle meet in a common point.

SUG'S.—First show that the intersection of two of the perpendiculars is equally distant from the three vertices of the triangle. Then that a line drawn from this point to the middle of the third side is perpendicular to it.

---

650. The three perpendiculars drawn from the angles of a triangle upon the opposite sides intersect in a common point.

SUG'S.—Draw through the vertices of the triangle lines parallel to the opposite sides. The proposition may then be brought under the preceding.

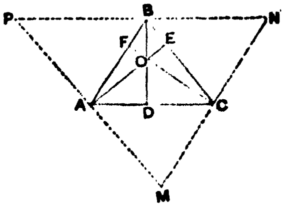


FIG. 364.

651. COR.—The following triangles are similar—viz., BOE, BDC, AOD, and AEC, each to each; also BOF, BDA, DOC, and CFA. Prove it.

---

652. **If from a point without a circle two secants be drawn, making equal angles with a third secant passing through the same point and the centre of the circle, the intercepted chords of the first two are equal.**

SUG.—Prove by revolving one part of the figure.

653. The sum of the alternate angles of *any* hexagon inscribed in a circle is four right angles.

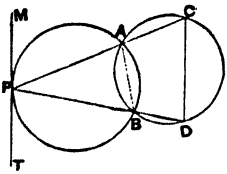


Fig. 365.

654. If two circles intersect in A and B, and from P, any point in one circumference, the chords PA and PB be drawn to cut the other in C and D, CD is parallel to a tangent at P.

---

655. If two lines intersect, two lines which bisect the opposite angles are perpendicular to each other.

---

656. The angle included by two lines drawn from a point within a triangle to the vertices of two of the angles, is greater than the third angle.

Sug's.—The demonstration may be founded on (*219*) or (*231*).

---

657. In a triangle whose angles are 90°, 60°, and 30°, show that the longest side is twice the shortest.

---

658. Lines which bisect the adjacent angles of a parallelogram are mutually perpendicular.

---

659. If from any point in the base of an isosceles triangle lines are drawn parallel to the sides, a parallelogram is formed whose perimeter is constant and equal to the sum of the two equal sides of the triangle.

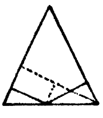


Fig. 366.

660. If from any point in the base of an isosceles triangle perpendiculars be drawn to the sides of the triangle, their sum is *constant* and equal to the perpendicular from one of the equal angles of the triangle upon the opposite side.

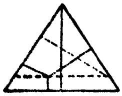


Fig. 367.

661. If from any point within an equilateral triangle, three perpendiculars be let fall upon the sides, their sum is constant and equal to the altitude of the triangle.

662. If from a fixed point without a circle two tangents be drawn terminating in the circumference, the triangle formed by them and any tangent to the included arc has a constant perimeter equal to the sum of the first two tangents.

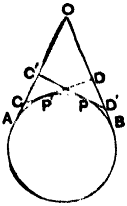


FIG. 368.

663. The sum of two opposite sides of a quadrilateral circumscribed about a circle, is equal to the sum of the other two.

664. If two opposite angles of a quadrilateral are supplementary, it may be circumscribed by a circumference.

665. The square described on the sum of two lines is equivalent to the sum of the squares on the lines, *plus* twice the rectangle of the lines.

SUG'S.—Be careful to give the construction fully, and show that the parts are rectangles, etc.

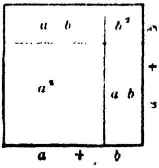


FIG. 369.

666. The square described on the difference of two lines is equivalent to the sum of the squares on the lines, *minus* twice the rectangle of the lines.

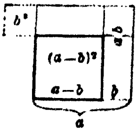


FIG. 370.

667. The rectangle of the sum and difference of two lines is equivalent to the difference of the squares described on the lines.

SCH.—The three preceding propositions are but geometrical conceptions and demonstrations of the algebraic *formulæ*, $(a + b)^2 = a^2 + 2ab + b^2$, $(a - b)^2 = a^2 - 2ab + b^2$, and $(a + b)(a - b) = a^2 - b^2$.

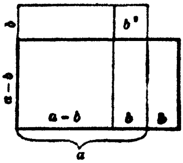


FIG. 371.

VARIOUS DEMONSTRATIONS OF THE PYTHAGOREAN PROPOSITION.

668. The square described on the hypotenuse of a right-angled triangle is equivalent to the sum of the squares described on the other two sides.

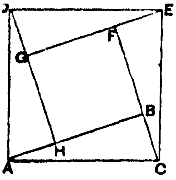


FIG. 372.

1st METHOD.—Let **ABC** be the given triangle, and **ACED** the square described on the hypotenuse. Complete the construction. Show that the four triangles are equal. The square **HF** is $(AB - BC)^2$. The student can complete the demonstration.

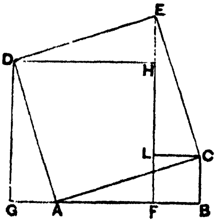


FIG. 373.

2d METHOD.—Let **ACED** be the square on the hypotenuse. Let fall the perpendiculars **EF**, **DG**, etc. Show that the three triangles are equal, and that **FD** and **LB** are the squares of the two sides **AB** and **BC**.

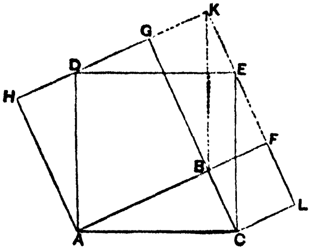


FIG. 374.

3d METHOD.—Let **BL** and **BH** be the squares on the sides. Produce **FL** and **HC** till they meet in **K**. Draw **DA** and **EC** perpendicular to **AC**, and draw **DE** and **KB**. Prove that **ACED** is a square, and also that the triangles **ABC**, **CLE**, **BFK**, **KBC**, **DKE**, and **AHD** are all equal to each other. The demonstration is then readily made.

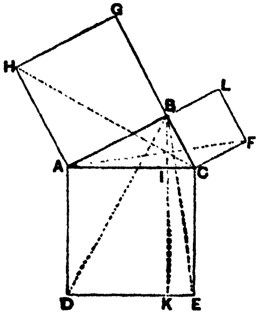
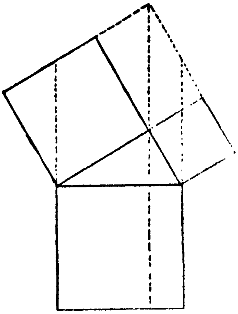


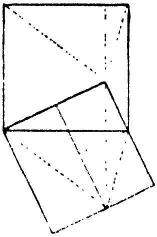
FIG. 375.

4th METHOD.—This is the demonstration usually given in our text-books. Drawing the squares on the three sides, let fall **BI** perpendicular to **AC** and produce it to **K**. Draw **BD**, **BE**, **HC** and **AF**. Show that the triangle **HAC** = **BAD**, and that the former is half the square **AG**, and the latter half the rectangle **AK**. Hence **AG** = **AK**. In like manner show that **LC** = **CK**.

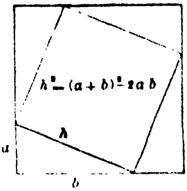
We will now give a few other figures by means of which the demonstration can be effected, and leave the student to his own resources in effecting it.



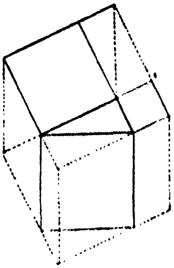
5th Method.



6th Method.



7th Method.



8th Method.

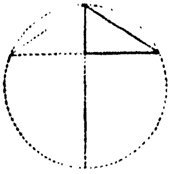


Fig. 377.

9th Method.—The truth of the theorem appears also as a direct consequence of (*360*).

669. In an oblique angled triangle the square of a side opposite an acute angle is equivalent to the sum of the squares of the other two sides diminished by twice the rectangle of the base, and the distance from the acute angle to the foot of the perpendicular let fall upon the base from the angle opposite.

Sug's.—It is to be shown that $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2AC \times DC$. Observe that $\overline{AD}^2 = (AC - DC)^2 = AC^2 + DC^2 - 2AC \times DC$. Whence, by a simple application of the preceding theorem, the truth of this becomes apparent.

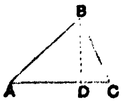


Fig. 378.

670. In an obtuse angled triangle the square of the side opposite the obtuse angle is equivalent to the sum of the squares on the other two sides, increased by twice the rectangle contained by the base and the distance from the obtuse angle to the foot of the perpendicular let fall from the angle opposite upon the base produced.

Sug.—The demonstration is analogous to the preceding, C being made obtuse in this case; whence $AD = AC + DC$, etc.

671. The following is an outline of a general demonstration covering the three preceding propositions:

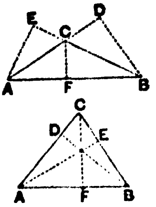


FIG. 379.

Letting AE, BD, and CF be the three perpendiculars from the angles upon the opposite sides, and observing that a circumference described on any side as a diameter passes through the feet of two of the perpendiculars, (***356***) and (***355***) readily give the following:

$$AB \times AF = AC \times AD = \overline{AC}^2 \pm AC \times CD,$$

$$\text{and } \underline{AB \times BF = BC \times BE = \overline{BC}^2 \pm BC \times CE};$$

adding, $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 \pm 2AC \times CD$ (or $2BC \times CE$), the + sign being taken when C is obtuse, and the − sign when C is acute. If C is right CE and CD become 0, whence $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

672. DEF.—The line drawn from any angle of a triangle to the middle of the opposite side is called a *medial line.*

673. The sum of the squares of any two sides of a triangle is equivalent to twice the square of the medial line drawn from their included angle, plus twice the square of half the third side.

SUG.—Proved by applying (***669, 670***).

674. The three medial lines of a triangle mutually trisect each other, and hence intersect in a common point.

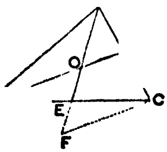


FIG. 380.

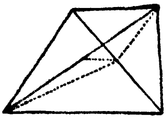
SUG'S.—To prove that $OE = \frac{1}{3}BE$, draw FC parallel to AD until it meets BE produced. Then the triangles AEO and FEC are equal (?); whence EF = OE. Also, BO = OF (?).

Having shown that $OE = \frac{1}{3}BE$, by a similar construction we can show that $OD = \frac{1}{3}AD$.

Finally, we may show that the medial line from C to AB cuts off $\frac{1}{3}$ of BE, and hence cuts BE at the same point as does AD.

ANOTHER DEM.—Lines through O parallel to the sides trisect the sides, etc.

. In any quadrilateral the sum of the squares of the sides is equivalent to the sum of the squares of the diagonals, plus four times the square of the line joining the centres of the diagonals.



381.

676. COR.—The sum of the squares of the sides of a parallelogram is equivalent to the sum of the squares of the diagonals.

677. In any quadrilateral which may be inscribed in a circle, the product of the diagonals is equal to the sum of the products of the opposite sides.

678. In any triangle the rectangle of two sides is equivalent to the rectangle of the perpendicular let fall from their included angle upon the third side, into the diameter of the circumscribed circle.

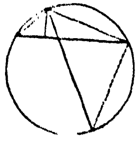


FIG. 382.

SUG.—This proposition is an immediate consequence of the similarity of two triangles in the figure.

679. COR.—The area of a triangle is equivalent to the product of its sides divided by twice the diameter of the circumscribed circle.

680. If there be an isosceles and an equilateral triangle on the same base, and if the vertex of the inner triangle is equally distant from the vertex of the outer one and from the ends of the base, then, according as the isosceles triangle is the inner or the outer one, its base angle will be ¼ of, or 2½ times the vertical angle.

681. Of all triangles on the same base, and having the same vertical angle, the isosceles has the greatest area.

SUG'S.—Describe a segment on the given base, which shall contain the given angle. The triangle on this base and having its vertex in the arc of the segment is the triangle to be considered.

682. Two triangles are similar, when two sides of one are proportional to two sides of the other, and the angle opposite to that side which is equal to or greater than the other given side in one, is equal to the homologous angle in the other.

ALGEBRAIC DEMONSTRATIONS.

3. The difference of the squares on any side of a regular pentagon and any side of regular decagon, inscribed in the same circle, is equivalent to the square of the radius.

SUG'S.—We will give the outline of what may be termed an *Algebraic Demonstration* of this proposition. This method is often the most convenient and expeditious. Letting p represent a side of the pentagon, d a side of the decagon, and r the radius, the student should be able to discover the following relations:

$$(1)\quad r : d :: d : r - d, \text{ or } r^2 - dr = d^2;$$

$$(2)\quad \sqrt{d^2 - \tfrac{1}{4}p^2} + \sqrt{r^2 - \tfrac{1}{4}p^2} = r.$$

From (2), $2r\sqrt{r^2 - \frac{1}{4}p^2} = 2r^2 - d^2 = r^2 + dr$, by substituting for d^2 its value from (1). Hence $4r^2 - p^2 = r^2 + 2dr + d^2$, or $3r^2 - 2dr = p^2 + d^2$. In this, substituting the value of dr as found in (1), we readily obtain $r^2 = p^2 - d^2$.

Q. E. D.

---

684. Demonstrate *algebraically* that the square on the sum of two lines, together with the square on the difference, is double the sum of the squares on the lines separately.

---

685. The sum of the squares of the three medial lines of a triangle is three-fourths of the sum of the squares of the sides.

---

686. The square of any side of a triangle is equivalent to twice the sum of the squares of the segments of the medial lines adjacent to its extremities, minus the square of the non-adjacent segment of the third medial line.

Deduced algebraically from the preceding.

---

687. The sum of the squares of the three greater segments of the medial lines of a triangle is equivalent to one-third the sum of the squares of the sides of a triangle.

Deduced algebraically from the preceding.

---

688. The lines from the vertices of a triangle to the points of tangency of the inscribed circle intersect in a common point.

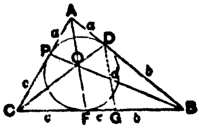


FIG. 382.

SUG'S. DG is parallel to AF, BD = BF = b, CF = CP = c, AD = AP = a, DG = d, FG = e. OF = $\frac{dc}{c + e}$, $d = \frac{\text{AF} \times b}{a + b}$, $e = \frac{ab}{a + b}$. $\therefore$ OF = AF × $\frac{bc}{a(b + c) + bc}$. In like manner we may find where PB intersects AF, by drawing through P a parallel to AF. This distance is found to be OF = AF × $\frac{bc}{a(b + c) + bc}$, a result which might have been anticipated, since b and c are similarly involved.

689. The area of a triangle, as expressed in terms of its sides, is *Area = the square root of the continued product of half the sum of the sides into this half sum minus each side separately.*

SUG'S.—We will give the outlines of both the Geometric and Algebraic demonstrations:

1st. GEOMETRIC DEMONSTRATION. CD = CB, CE = CA, and through F, the centre of DA, HG is drawn parallel to AB. With F as a centre, and FH as a radius, a circumference passes through G (?). CN is perpendicular to AE and passes through H (?).

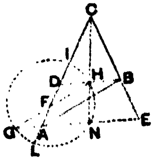


FIG. 384.

Now CF = ½ (AC + CB), and FL = ½AB (?).

Hence CL = ½(AC + CB + AB) = ½S, S being the sum of the sides;

Hence, also, DL = AI = ½S − CB, CI = ½S − AB, and AL = ½S − AC.

Again, CN × AN = area ACE;

and HN × AN = area ABE;

Subtracting, AN (CN − HN) = AN × CH = area ACB (1).

Once more, CH × DH = area CDB;

and HN × DH = area ADB.

Adding, DH (CH + HN) = CA × CN = area ACB (2).

Multiplying (1) and (2), we have

$$CA \times AN \times CH \times CN = (\text{area } ACB)^2.$$

But CN × CH = CL × CI, and CA × AN = AL × AI = AL × DL.

Therefore, *area* ACB $= \sqrt{CL \times CI \times AL \times DL} =$

$$\sqrt{\tfrac{1}{2}S\,(\tfrac{1}{2}S - AB)\,(\tfrac{1}{2}S - AC)\,(\tfrac{1}{2}S - CB)}.$$

2d. ALGEBRAIC DEMONSTRATION. From the right angled triangles BCD and ACD, we find $m = \dfrac{a^2 - b^2 + c^2}{2c}$.

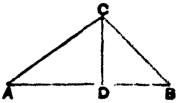


FIG. 385.

Whence $p = \sqrt{a^2 - \left(\dfrac{a^2 - b^2 + c^2}{2c}\right)^2}$, and

$$\text{area } ABC = \frac{c}{2}\sqrt{a^2 - \left(\frac{a^2 - b^2 + c^2}{2c}\right)^2}$$

$$= \tfrac{1}{4}\sqrt{-a^4 + 2a^2b^2 + 2a^2c^2 - b^4 + 2b^2c^2 - c^4}$$

$$= \tfrac{1}{4}\sqrt{(a + b + c)(a + b - c)(a - b + c)(-a + b + c)}$$

$$= \sqrt{\tfrac{1}{2}s(\tfrac{1}{2}s - c)(\tfrac{1}{2}s - b)(\tfrac{1}{2}s - a)}.$$

690. From any point in the plane of a circle the greatest and least distances to the circumference are measured on the line passing through the centre.

Sug's.—There are three cases :—1st. When the point is without the circle. 2d. When the point is within. 3d. When the point is in the circumference.

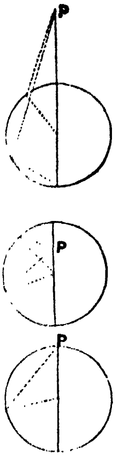


Fig. 386.

691. From any point except the centre of a circle, two, and only two, equal lines can be drawn to the circumference.

Sug.—This is a direct consequence of (*181*, *182*).

692. If two opposite sides of a parallelogram be bisected, straight lines from the points of bisection to the opposite vertices will trisect the diagonal.

693. The feet of two perpendiculars let fall from two given points upon a given line are equally distant from the middle of the line joining the points.

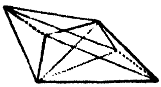


Fig. 387.

694. Two quadrilaterals are equivalent when their diagonals are respectively equal, and form equal angles.

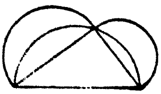


Fig. 388.

695. If, on the hypotenuse and sides of a right angled triangle, semicircumferences be described. that upon the hypotenuse passing through the vertex, the sum of the crescents thus formed will be equal to the area of the triangle.

696. The bisectors of any two exterior angles of a triangle meet in a point which is the centre of a circle, to which one side of the triangle and the other two produced are tangents.

These circles are called the *escribed circles.*

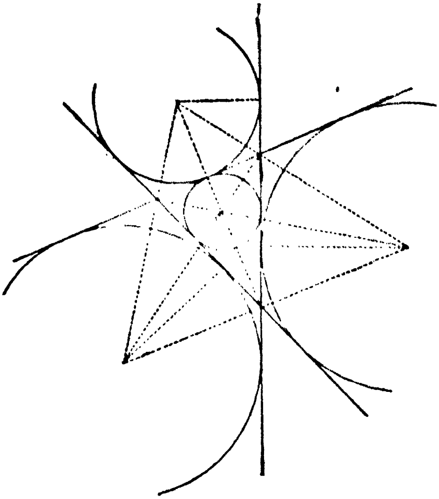


Fig. 389.

THE NINE POINTS CIRCLE.

697. In any triangle the centres of the THREE sides, the feet of the THREE perpendiculars from the vertices upon the opposite sides or sides produced, and the THREE middle points of the distances from the vertices to the common intersection of the perpendiculars, are NINE points in the circumference of one and the same circle; the centre of this circle is at the middle of the line joining the centre of the circumscribed circle and the common intersection of the perpendiculars; and the radius is half the radius of the circumscribed circle.

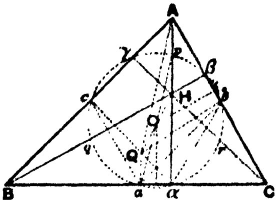


Fig. 390.

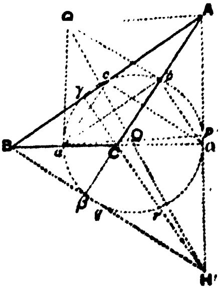


Fig. 391.

SUG'S.—The student will do well to confine his attention in the first instance to the first figure, and after he sees the demonstration in this case—*i. e.*, when the perpendiculars fall within, to trace it in the case of an obtuse angled triangle, in which the perpendiculars fall on the sides produced

1st. To show that the circle which passes through a, b, and c, also passes through α, we show the following relations among the angles: $cab = cAb =$

$+ \alpha Ab = cab$. Hence, the vertices a and α are in the same circumference. In like manner we show that β and γ are in the circumference passing through a, b, and c.

2d. Considering one of the partial triangles as BHC, α, β, and γ are the feet of the three perpendiculars from *its* vertices upon one of its sides and the prolongation of the other two. Therefore, by the first part r and q are the middle points of CH and BH. Considering either of the other partial triangles we find p the centre of AH.

3d. $a\alpha$ and $b\beta$ being chords of the nine points circle, O is its centre, and letting Q be the centre of the circumscribed circle, we may readily show that O is *in* QH, and also is at its middle point.

4th. Drawing aO, and producing it, we may show that it intersects AH in p, and hence $p\mathrm{H} = \mathrm{A}p = \mathrm{Q}a$, and $\mathrm{AQ}ap$ is a parallelogram. Therefore $\mathrm{O}p = \frac{1}{2}pa = \frac{1}{2}\mathrm{QA}$.

698. COR.—The middle points of the three lines joining the centres, two and two, of the escribed circles of a triangle, and the middle points of the three lines joining the centres of the escribed circles with the centre of the inscribed circle, are six points in the circumference of the circle circumscribed about the same triangle.

699. If one triangle be inscribed in another, the circumferences circumscribing the three exterior triangles thus formed intersect in a common point.

SUG.—The demonstration is founded on the property of the opposite angles of an inscribed quadrilateral. The construction lines extend from the vertices of the inscribed triangle to the intersection to be examined.

700. The difference between the hypotenuse and the sum of the other two sides of a right angled triangle is equal to the diameter of the inscribed circle.

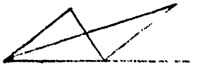


FIG. 392.

701. If from the extremities of any side of a triangle two lines be drawn, one bisecting an interior and the other an exterior angle, these lines will meet if sufficiently produced, and their included angle will be half the third angle of the triangle.

702. An inscribed equilateral triangle is one-fourth the circumscribed equilateral triangle about the same circle.

703. The three altitudes of a triangle are to each other inversely as the sides upon which they fall.

704. The bisectors of the angles included by the opposite sides of an inscribed quadrilateral intersect at right angles.

SUG.—By means of (***214***) show that FC + CE + HA + AC = 180°. Whence FOE = 90°.

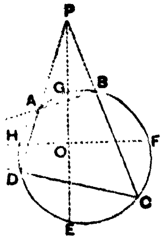
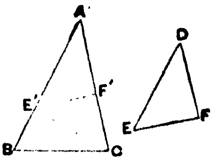


FIG. 393.

705. Two triangles which have an angle in each equal, are to each other as the rectangle of the sides including the equal angle.

SUG'S. A and D being equal, we are to show that ABC : DEF :: AB × AC : DE × DF. Take AE′ = DE, AF′ = DF, and draw E′F′. Now from the facts that the triangles AE′F′ and DEF are equal, and that triangles of the same altitudes are to each other as their bases, the proposition is proved.



706. If of the four triangles into which the diagonals divide a quadrilateral, two opposite ones are equivalent, the figure is a trapezoid.

---

707. The difference between the angles which a bisector in a triangle makes with the side to which it is drawn, is equal to the difference of the angles of the triangle including this side.

708. If any number of equal right lines radiate from a common point, making equal consecutive angles, and any line be drawn through the common point, the sum of the perpendiculars upon this line from the extremities of the radiating lines on one side, is equal to the sum of those on the other side.

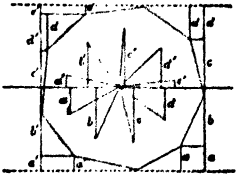


FIG. 395.

709. COR.—In any regular polygon, the sum of the perpendiculars let fall from the vertices on the one side of any line passing through the centre, is equal to the sum of those let fall from the vertices on the other side.

710. If the sum of two opposite sides of a quadrilateral is equal to the sum of the other two opposite sides, a circle may be inscribed in it.

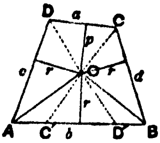


Fig. 896.

Sug's.—Bisect any two adjacent angles, as A and B. Then are the perpendiculars r, r, r, equal (?); and it remains to be shown that the perpendicular $p = r$. Take AD′ = AD, and BC′ = BC, and draw OD′ and OC′. Since $a + b = c + d$, and $C'D' = b - (b - c) - (b - d)$, $C'D' = a$, and the triangles DOC and D′OC′ are equal. Hence $p = r$.

711. If two planes are parallel, any right line which pierces one, pierces the other also.

Sug.—Proof based on (***410***).

712. If two planes are parallel, any plane which intersects one, intersects the other also, and the lines of intersection are parallel.

713. Cor.—Two planes which are parallel to a third, are parallel to each other.

714. A plane which is perpendicular to a line of another plane, or to a line parallel to that plane, is itself perpendicular to the latter plane.

---

715. If a straight line is perpendicular to a plane, any line parallel to the plane is perpendicular to the first line.

Sug.—Two lines in space which are not in the same plane, are said to make the same angle with each other as two lines respectively parallel to them and both lying in one plane.

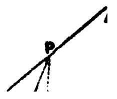
---

716. In order that a straight line be perpendicular to a plane, it is sufficient that it be perpendicular to two lines not parallel to each other, and situated in the plane or parallel to it.

717. If two right lines in space are perpendicular to each other (not necessarily intersecting), their projections on a plane parallel to either line are perpendicular to each other.

Sug.—The *Projection* here referred to is that which is called the *Orthographic Projection*. The proposition is not generally true of the *Perspective Projection*, *i. e.*, the spaces which the lines (considered as material) would appear to occupy if they were placed between the eye and the plane. (See Ex. 8, page 174, Part II.)

718. The angle of inclination (***392***, Part II.) of a line oblique to a plane, is less than the angle included between this line and any line of the plane, except its projection, which passes through the point in which the first line pierces the plane.



Sug. **BD** being the projection of **AB**, **ABD** < ∠**ABD′**, **BD′** being any line other than **BD**, passing through **B**.

Fig. 397.

719. Between any two lines not in the same plane, one line, and only one, can be drawn, which shall be perpendicular to both the given lines.

Sug's.—Pass a plane through one of the lines parallel to the other; and through the other line pass a plane perpendicular to the first plane.

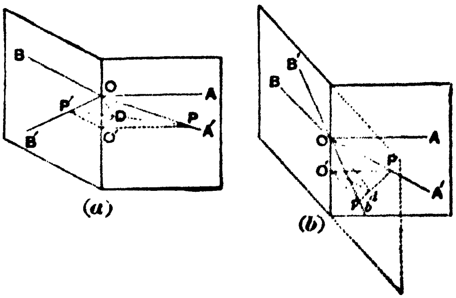
720. In a warped quadrilateral, *i. e.*, one whose sides do not all lie in the same plane, the middle points of the sides are in one plane, and are the vertices of the angles of a parallelogram.

Sug.—Conceive the planes of two opposite angles of the quadrilateral, the intersection of which will be a diagonal of the given quadrilateral.

721. A line being given in a plane, one plane can be drawn including the given line and perpendicular to the first plane, and only one. Hence all right diedrals are equal.

Sug.—Demonstration similar to

722. The plane angle formed by drawing two lines in the faces of a diedral, from a common point in the edge, is less than the measure of the diedral if the angle is acute, and lines lie on the same side of the plane of the measure and are equally inclined to it, and greater if they lie on opposite sides.



In general, if the diedral is acute, the limits of the varying angle are 0 and 90° when both the lines lie on the same side of the plane of the measuring angle, and 180° when they lie on opposite sides. If the diedral is obtuse the limits are 0 and the angle itself when the lines lie on the same side of the measure, and 90° and 180° when they lie on opposite sides.

723. If the projections of a line in the two faces of a diedral are straight, the line is a straight line.

SUG.—Proof based on (*386*).

724. If from the vertex of a triedral a line be drawn at pleasure within the triedral, the sum of the plane angles formed by this line and any two edges is less than the sum of the facial angles formed by the other edge and these two.

725. If through a point in space two lines be drawn parallel to a given plane, and through the same point two planes be passed respectively perpendicular to the two lines, the intersection of these two planes will be perpendicular to the given plane.

726. The three planes which bisect the three diedrals of a triedral intersect in a common line.

727. In any convex polyedral, the sum of the diedrals is greater than the sum of the angles of a polygon having the same number of sides that the polyedral has faces.

SUG.—Proof based upon (*722*).

728. DEF.—***A Polyedron*** is a solid bounded by plane surfaces. A *Regular Convex Polyedron* is a polyedron whose faces are all equal regular polygons, and each of whose solid angles is convex outward, and is enclosed by the same number of faces.

729. There are five and only five regular convex polyedrons—viz.: *The Tetraedron*, whose faces are four equal equilateral triangles; *The Hexaedron*, or *Cube*, whose faces are six equal squares; *The Octaedron*, whose faces are eight equal equilateral triangles; *The Dodecaedron*, whose faces are twelve equal regular pentagons; and *The Icosaedron*, whose faces are twenty equal equilateral triangles.

DEM.—We demonstrate this proposition by showing—1st, that such solids can be constructed; and 2d, that no others are possible.

The Regular Tetraedron.—Taking three equal equilateral triangles, as **ASB**, **ASC**, and **BSC**, it is possible to enclose a solid angle, as **S**, with them, since the sum of the facial angles is (what?) (PART II., *436*). Then, since **AC** = **AB** = **CB** (?), considering **ACB** the fourth face, we have a regular polyedron whose four faces are equilateral triangles.

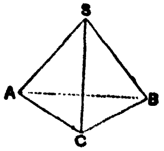


FIG. 399.

The Regular Hexaedron or *Cube.*—This is a familiar solid, but for purposes of uniformity and completeness we may conceive it constructed as follows: Taking three equal squares, as **ASCB**, **CSED**, and **ASEF**, we can enclose a solid angle, as **S**, with them (?). Now, conceive the planes of **CB** and **CD**, **AB** and **AF**, **EF** and **ED** produced. The plane of **CB** and **CD** being parallel to **ASEF** (?) will intersect the plane of **EF** and **ED** in **HD** parallel to **FE** (?). In like manner **FH** can be shown parallel to **ED**, **BH** to **CD**, and **HD** to **BC**. Hence the solid has for its faces six equal squares.

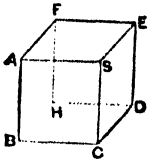


FIG. 400.

The Regular Octaedron.—At the intersection **P**, of the diagonals of a square, **ABCD**, erect a perpendicular **SP** to the plane of the square, and making **SP** = **AP** (half of one of the diagonals) draw **SA**, **SD**, **SC**, and **SB**. Making a similar construction on the other side of the plane **ABCD**, we have a solid having for faces eight equal equilateral triangles.

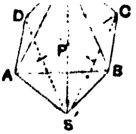


FIG. 401.

The Regular Dodecaedron.—Taking twelve equal regular pentagons, it is evident that we may group them in two sets of six each, as in the figure. Thus, around **O** we may place five, forming 5 triedrals at the vertices of **O**. These triedrals are possible, since the sum of the facial angles enclosing each is $3\frac{3}{5}$ right angles (?) —*i. e.*, between 0 and 4 right angles (PART II., *436*). In like manner the other 6 may be grouped by placing 5 of them about **O′**. Now, conceiving the *convexity* of the group **O** in front and the *concavity* of group **O′**, we may place the two together so as to inclose a solid. Thus, placing **A** at *b*, the three faces 5, 6, 1, will enclose a triedral, since the diedral included by 5 and 1 is the diedral of such a triedral. Then will vertex **B** fall at *c*, and a like triedral will be formed at that point, and so of all the other vertices. Hence we have a polyedron having for faces 12 equal regular pentagons.

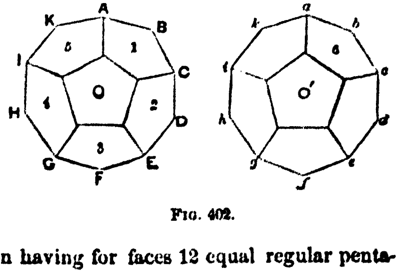


FIG. 402.

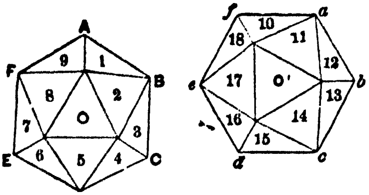


FIG. 403.

The Regular Icosaedron.—Taking 20 equal equilateral triangles, they can be grouped in two sets, as in the figure, in a manner altogether similar to the preceding case. The solid angles in this case are included by 5 facial angles whose sum is $3\frac{1}{3}$ right angles (?), which is a possible case (PART II., ***436***). As before, conceiving the *convexity* of group O in front, and the *concavity* of O′, we can place them together by placing A at *a*, thus enclosing a solid angle with 5 faces, whence B will fall at *b*, etc. Thus we obtain a solid with 20 equal equilateral triangles for its faces.

That there can be no other regular polyedrons than these 5 is evident, since we can form no other convex solid angles by means of regular polygons. Thus, with equilateral triangles (the simplest polygon) we have formed solid angles with 3 faces (the least number possible), as in the tetraedron; with 4, as in the octaedron; and with 5, as in the icosaedron. Six such facial angles cannot enclose a solid angle, since their sum is four right angles (?), and much less any greater number. Again, with squares (the next most simple polygon) we have formed solid angles with 3 faces as in the hexaedron, and can form no other, for the same reason as above. With regular pentagons we can only enclose a triedral, as in the dodecaedron, for a like reason. With regular hexagons we cannot enclose a solid angle (?), and much less with any regular polygon of more than six sides.

SCH.—Models of the regular polyedrons are easily formed by cutting the following figures from cardboard, cutting half-way through the board in the dotted lines, and bringing the edges together as the forms will readily suggest.

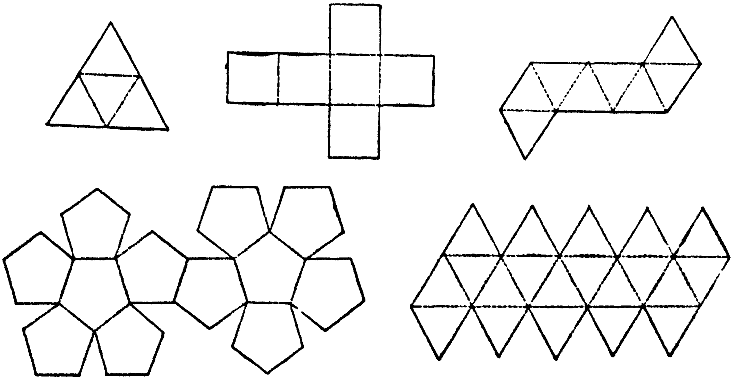


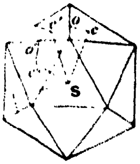
FIG. 404.

730. Any regular polyedron is inscriptible and circumscriptible by a sphere.

SUG'S.—From the centres of any two adjacent faces, as c and c', let fall perpendiculars upon the common edge, and they will meet it in the same point o (?). The plane of these lines will be perpendicular to this edge (?), and perpendiculars to these faces from their centres, as $c\mathrm{S}$, $c'\mathrm{S}$, will lie in this plane (?), and hence will intersect at a point equally distant from these faces.

In like manner $c''\mathrm{S} = c'\mathrm{S}$, and the point S can be shown to be equally distant from all of the faces, and is therefore the centre of the inscribed sphere.

Joining S with the vertices, we can readily show that S is also the centre of the circumscribed sphere.



Fig

731. Show that a, being the edge of a regular tetraedron, its volume is $\frac{a^3\sqrt{2}}{12}$.

---

732. DEF.—*A Truncated Prism* is one whose upper and lower bases are not parallel.

733. The volume of a truncated triangular prism is equal to the sum of the volumes of three pyramids whose common base is the lower base of the prism, and whose vertices are the angles of the upper base.

SUG'S.—Let $b\mathrm{D}$, $c\mathrm{D}'$, and $a\mathrm{D}''$ be perpendicular to the lower base. Volume of b-ABC is $\frac{1}{3}\, b\mathrm{D} \times \mathrm{ABC}$. Volume a-$b\mathrm{C}c$: volume b-ABC : : $cb\mathrm{C}$: $b\mathrm{BC}$: : $c\mathrm{C}$: $b\mathrm{B}$: : $c\mathrm{D}'$: $b\mathrm{D}$. ∴ Volume a-$b\mathrm{C}c$ $= \frac{1}{3}c\mathrm{D}' \times \mathrm{ABC}$. In a similar manner volume : $\frac{1}{3}a\mathrm{D}'' \times \mathrm{ABC}$.

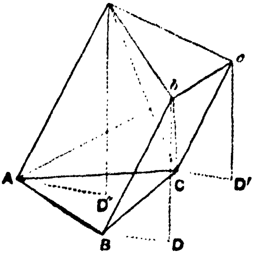


FIG. 406.

734. COR.—The volume of a prism, one of whose bases is a right section and the other an oblique section, is the product of the right section into the arithmetical mean of its edges.

SUG'S.—The volume of abc-ABC is as shown above $\mathrm{ABC}\left(\frac{b\mathrm{D} + c\mathrm{D}' + a\mathrm{D}''}{3}\right)$. But if ABC is a right section, $b\mathrm{D} = b\mathrm{B}$, $c\mathrm{D}' = c\mathrm{C}$, and $a\mathrm{D}'' = a\mathrm{A}$. Hence the volume is $\mathrm{ABC}\left(\frac{b\mathrm{B} + c\mathrm{C} + a\mathrm{A}}{3}\right)$.

735. The volume of any polyedron having for its bases any two polygons whatever, situated in parallel planes, and for lateral faces trapezoids, is the product of $\frac{1}{6}$ the distance between the bases into the sum of the two bases plus 4 times a section midway between the bases; or $V = \frac{H}{6}(B + B' + 4B'')$, in which H is the distance between the bases, B and B' the bases, and B'' a section midway between the bases.

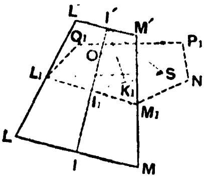


FIG. 407.

DEM.—Let $L_1 M_1 N_1 P_1 Q_1$ be the section of such a polyedron midway between its bases, and S any point in this section. Joining S with the vertices of the polyedron, we divide the solid into as many pyramids as it has faces. The volumes of the two which have B and B' for their bases are evidently $\frac{1}{6}H \times B$, and $\frac{1}{6}H \times B'$. It remains to find the volume of the others.

Let $LML'M'$ be a lateral face corresponding to L_1M_1 and SO a perpendicular from S upon this face. Draw $l'l$ through O perpendicular to LM, and consequently to $L'M'$. Take $l'K_1$ perpendicular to the plane section, whence $l'K_1 = \frac{1}{2}H$. Now the volume of the pyramid having $L'M'LM$ for its base and S for its vertex is $L_1M_1 \times 2l'l_1 \times \frac{1}{3}SO$. But $l'l_1 \times SO = Sl_1 \times l'K_1$ (?); whence the volume of this pyramid is $\frac{2}{3}L_1M_1 \times Sl_1 \times l'K_1 = \frac{1}{3} \times 2SL_1M_1 \times l'K_1 = \frac{1}{3}l'K_1 \times 4SL_1M_1 = \frac{1}{6}H \times 4SL_1M_1$. In like manner the volume of the pyramid having for its base the face in which M, N, is situated, can be shown to be $\frac{1}{6}H \times 4SM_1N_1$ and similarly of all the others. Whence the whole volume is $\frac{1}{6}H(B + B' + 4B'')$.

736. COR.—The proposition is equally true when some or all of the lateral faces are triangles; *i. e.*, when one base has more sides than the other.

SCH.—The preceding propositions are of much value in calculating earthwork.

737. If we cut a pyramid by a plane parallel to its base, a second pyramid is formed similar to the first.

738. Two triangular pyramids are similar whenever they have an equal diedral angle contained between faces, similar each to each, and similarly placed.

739. Two polyedrons composed of the same number of tetraedrons, similar each to each, and similarly disposed, are similar.

740. All regular polyedrons of the same number of faces are similar solids.

741. The intersection of the surfaces of two spheres is the circumference of a circle whose plane is perpendicular to the line which joins their centres.

742. Through any four points not in the same plane one sphere may be made to pass, and only one.

SUG'S.—The four points may be considered as the vertices of a tetraedron. Conceive perpendiculars drawn to the triangular faces from the intersections of lines drawn in these faces perpendicular to the sides at their middle points. These perpendiculars will meet at a common point (?), which is the centre of the circumscribed sphere (?).

[The student should show why *only* one sphere can be circumscribed.]

743. COR. 1.—The four perpendiculars erected at the centres of the circles circumscribing the faces of a tetraedron intersect at a common point.

744. COR. 2.—The six planes, perpendicular to the six edges of a tetraedron at their middle points, intersect at the centre of the circumscribed sphere.

745. One sphere and only one may be inscribed in any tetraedron.

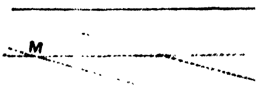
SUG.—Bisect the diedrals with planes.

746. The angle included by any two curves intersecting on the surface of a sphere, is equal to the angle included by the arcs of two great circles passing through the point of intersection, and whose planes produced include the tangents to the curves at their intersection.

SECTION II.

PROBLEMS IN SPECIAL OR ELEMENTARY GEOMETRY.

747. To bisect the angle formed by two lines whose intersection is inaccessible.



SUG.—M and N are points in the bisector.

FIG. 408.

748. To pass a circumference through three points, not in the same straight line, when the radius is so long as to render the ordinary method impracticable.

SUG.—Let **A**, **B**, and **C** be the three points; then are **M** and **N** other points in the same circumference.

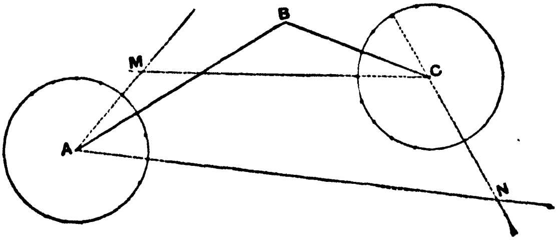


FIG. 409.

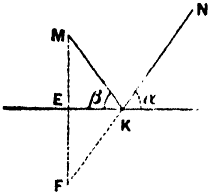


FIG. 410.

749. From two given points on the same side of a line given in position, to draw two lines which shall meet in that line and make equal angles with it.

SUG.—If α and β are equal, what is the relation of **ME** to **EF**?

750. To construct an isosceles triangle with a given base and vertical angle.

SUG.—See Prob. 4, p. 102.

1. To trisect a right angle.

SUG.—What is the value of an angle of an equilateral triangle?

752. Given the perpendicular of an equilateral triangle, to construct the triangle.

753. Given the diagonal of a square, to construct it.

754. To construct an isosceles triangle, so that the base shall be a given line, and the vertical angle a right angle.

755. Given the sum of the diagonal and a side of the square, to construct it.

SUG.—What are the values of α and β respectively?

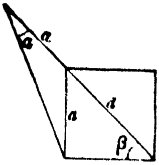


FIG. 411.

756. To construct a triangle when the altitude, the vertical angle, and one of the sides are given.

757. To construct a triangle when the sum of the three sides and the angles at the base are given.

SUG'S.—MN being the sum of a, b, and c, what are the angles M and N as compared with the given angles α and β?

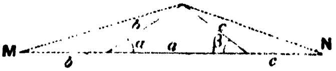


FIG. 412.

758. In a right angled triangle the perimeter, and the perpendicular from the right angle upon the hypotenuse being given, to construct the triangle.

SUG'S.—DE is equal to the perimeter, DBE is an angle of 135°, and FE is the perpendicular on the hypotenuse. ABC is the required triangle. Let the student give the solution in full, and the proof.

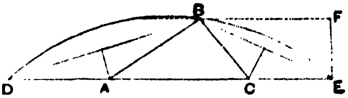


FIG. 413.

759. From two given points on the same side of a given line, to draw two equal straight lines which shall meet in the same point of the line.

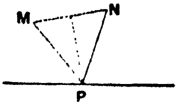


FIG. 414.

760. To pass a circumference through two given points, which shall have its centre in a given line.

761. To construct a quadrilateral when three sides, one angle, and the sum of two other angles are given.

SUG'S.—What is the fourth angle? When two sides and their included angle are known, there will be two cases, according as the two angles whose sum is known are adjacent to each other or opposite. In the latter case we have to describe a segment on a diagonal, which will contain the fourth angle. For the third case see Ex. 13, page 136.

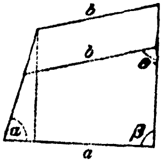


FIG. 415.

762. To construct a quadrilateral when three angles and two opposite sides are given.

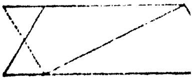


FIG. 416.

763. To bisect a trapezoid by a line drawn from one of its angles.

764. In a given circle, to inscribe a triangle equiangular with a given triangle.

SUG.—How does an angle at the centre compare with one inscribed in the same segment?

765. To describe three circles of equal diameters which shall touch each other.

766. In an equilateral triangle, to inscribe three equal circles which shall touch each other and the three sides of the triangle.

767. To describe a circle of given radius touching the two sides of a given angle.

SUG.—How far is the centre from each line?

768. To describe a circumference which shall be embraced between two parallels and pass through a given point within the parallels.

SUG.—In what line is the centre? How far from the given point?

769. To describe a circle with a given radius, which shall pass through a given point and be tangent to a given line.

770. To find in one side of a triangle the centre of a circle which shall touch the other two sides.

---

771. Through a given point on a circumference, and another given point without, to describe a circle touching the given circumference.

Sug.—Consider in what two lines the centre must lie.

772. In the diameter of a circle produced, to determine the point from which a tangent drawn to the circumference shall be equal to the diameter.

Sug.—What is the relation between the radius, the required tangent, and the distance from the centre to the intersection of the produced diameter and the required tangent?

773. To describe a circle of given radius, touching two given circles.

774. In a given circle, to inscribe a right angle, one side of which is given.

775. In a given circle, to construct an inscribed triangle of given altitude and vertical angle.

776. To inscribe a square in a given right-angled isosceles triangle, one side being in the hypotenuse.

777. To inscribe a square in a given quadrant of a circle, the vertex of an angle being at the centre.

778. To find the centre of a circle in which two given lines meeting in a point shall be a tangent and a chord.

779. To describe a circumference which shall pass through a given point and be tangent to a given line at a given point.

780. To bisect a quadrilateral by a line drawn from one of its angles.

Sug.—The demonstration is based upon the principle that triangles having equal bases and equal altitudes are equivalent.

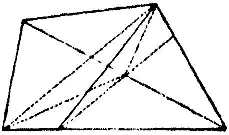


Fig. 418.

781. Through a given point situated between the sides of an angle, to draw a line terminating at the sides of the angle, and in such a manner as to be bisected at the point.

SUG.—Conceive the point as situated in the third side of a triangle of which the two given lines are the other two.

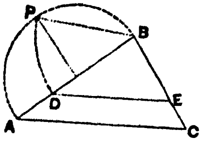


FIG. 419.

782. To draw a line parallel to the base of a triangle so as to divide the triangle into two equivalent parts.

SUG. $\overline{PB}^2 = \overline{DB}^2 = \frac{1}{2}\overline{AB}^2$. See (***344, 362***).

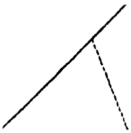


FIG. 420.

783. To construct a square when the difference between the diagonal and a side is given.

SUG.—Consider the angles.

784. To determine the point in the circumference of a circle from which chords drawn to two given points shall have a given ratio.

SUG.—Draw a chord dividing the chord joining the given points in the required ratio, and bisecting one of the subtended arcs.

785. To bisect a given triangle by a line drawn from one of its angles.

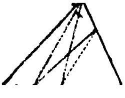


FIG. 421

786. To bisect a given triangle by a line drawn from a given point in one of its sides.

787. In the base of a triangle find the point from which lines extending to the sides, and parallel to them, will be equal.

788. To construct a parallelogram having the diagonals and one side given.

---

789. To construct a triangle when the three altitudes are given.

SUG.—What is the relation of the perpendiculars to the sides upon which they fall? If a triangle can be formed with the perpendiculars as sides, how will it compare with the first triangle? How proceed when the perpendiculars will not form a triangle?

---

790. What is the area of the sector whose arc is 50°, and whose radius is 10 inches?

---

791. To construct a square equivalent to the sum, or to the difference of two given squares.

---

792. To divide a given straight line in the ratio of the areas of two given squares.

---

793. To construct a triangle, when the altitude, the line bisecting the vertical angle, and the line from the vertex to the middle of the base are given.

SUG.—The centre of the circle circumscribing the required triangle is in the perpendicular to the base at its middle point; and the intersection of this perpendicular and the bisectrix is a point in this circumference.

Show that the bisector always lies between the perpendicular and the medial line.

794. Through a given point, draw a line such that the parts of it, between the given point and perpendiculars let fall on it from two other given points, shall be equal.

What would be the result, if the first point were in the straight line joining the other two?

795. From a point without two given lines, to draw a line such that the part intercepted between the given lines shall be equal to the part between the given point and the nearest line.

SUG.—Produce the lines till they meet, if necessary. Draw a line through the given point parallel to one of the lines, and produce it till it meets the other.

796. Given one angle, a side adjacent to it, and the difference of the other two sides, to construct the triangle.

QUERIES.—How if $b > a$? How if B is obtuse?

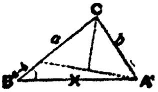


FIG.

---

797. To pass a circumference through two given points, having its centre in a given line.

798. To draw a line parallel to a given line and tangent to a given circumference.

Sug.—Draw a diameter perpendicular to the given line.

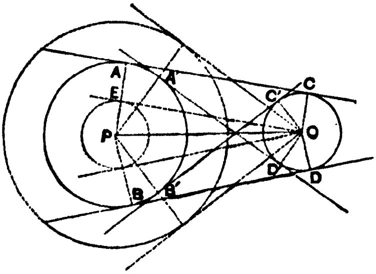


FIG. 423.

799. To draw a common tangent to two given circles.

Sug.—1st Method.—There are two sets of tangents, AC, BD, and A'D', B'C'. For the first, observe that if $PE = AP - OC$, OE is parallel to AC, etc.

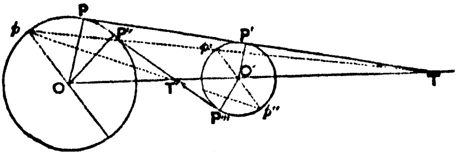


FIG. 424.

2d Method. pO, $p'O'$ being parallel to each other, $pp'T$ gives the intersection of the tangent with the line passing through the centres, since

$$pO : p'O' :: OT : O'T, \text{ or } pO - p'O' : p'O' :: OO' : O'T.$$

$$\text{Also, } PO - P'O' : P'O' :: OO' : O'T.$$

Hence O'T is constant for all positions of the parallel radii. Prove that if the parallel radii are on different sides of the line joining the centres, T' is the point where the internal tangent cuts OO'.

Queries.—How many tangents can be drawn—1st. When the circles are external one to the other; 2d. When they are tangent externally; 3d. When they intersect each other; 4th. When tangent internally; 5th. When one lies within the other?

800. To describe a circle tangent to a given circumference and also to a given line at a given point.

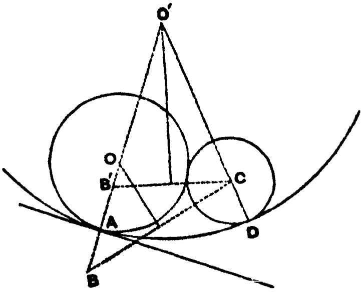


FIG. 425.

Sug's.—There may be two cases—1st. When the given circle is exterior to the one sought; and 2d. When it is interior. In either case the centre of the required circle is in the perpendicular AO'. In the former case, O, the centre of the required circle, is at $r + r'$ from C; and in the latter O' is at $r - r'$ from C. AO $= r$, and **CD = AB = AB' $= r'$.**

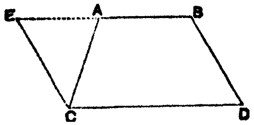


FIG. 426

801. To construct a trapezoid when the four sides are given.

SUG.—Knowing the difference between the two parallel sides, we may construct the triangle **AEC**, and hence the trapezoid.

802. On a given line, to construct a polygon similar to a given polygon.

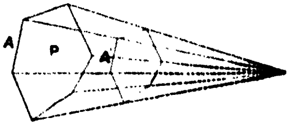


FIG. 427.

SUG'S.—One method may be learned from (***90***). Ex. 8, page 152, furnishes another method. The following is an elegant method: To construct on **A'** homologous with **A**, a polygon similar to **P**. Place **A'** parallel to **A**, and the figure will suggest the construction.

---

803. To pass a plane through a given line and tangent to a given sphere.

SUG'S.—Pass a plane through the centre of the sphere and perpendicular to the given line. Through the point of intersection and in this secant plane draw tangents to the great circle in which the secant plane intersects the surface of the sphere. The points of tangency will be the points of tangency of the required planes (?), of which there are thus seen to be two.

---

804. DEF.—*A Tangent Plane* to a cylindrical or conical surface is a plane which contains an element of the surface, but does not cut the surface. The element which is common to the surface and the plane is called the *Element of Contact.*

805. To pass a plane through a given point and tangent to a given cylinder of revolution.

SUG'S.—1st. When the point is in the surface of the cylinder. Through the point draw an element of the cylinder, by passing a line parallel to the axis, or to any given element. Through the same point pass a plane perpendicular to this element, making a right section (a circle). To this circle draw a tangent. The plane of the element and tangent is the tangent plane required. [The student should prove that any point in the plane affirmed to be tangent, not in the element passing through the given point, is without the cylinder.]

2d. When the given point is without the cylinder. Pass a plane through the given point perpendicular to the axis of the cylinder, thus making a right section of the cylinder (a circle). In this secant plane draw tangents to the section. Through the points of contact of these tangents draw elements of the cylinder. These elements are the elements of contact of the tangent planes. Hence planes passing through them and the given point are the tan-

gent planes required. [The student should remember that this is but an *outline*, and be careful to fill it up, giving the proof.]

806. To pass a plane through a given point and tangent to a conical surface of revolution.

807. To find, with the compasses and ruler, the radius of a material sphere whose centre is inaccessible.

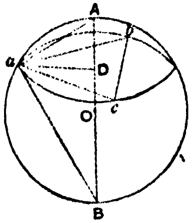


FIG. 428.

SUG'S.—With one point of the compasses at any point in the surface, as A, trace a circle of the sphere, as *acb*. The chord A*a* is measured by the distance between the compass points. In like manner measure three other chords, as *ac*, *ab*, and *bc*. Draw a plane triangle having these chords for its sides, and circumscribe a circle about it. Thus *a*D is found. Knowing *a*A, and *a*D, and remembering that A*a*B is right angled at *a*, the triangle A*a*B can be drawn in a plane (?), whence AO becomes known.

SECTION III.

APPLICATIONS OF ALGEBRA TO GEOMETRY.

808. The mathematical method which is called technically *Applications of Algebra to Geometry* consists in finding, by means of equations, the numerical values of the unknown parts of a geometrical figure, when a sufficient number of the parts are given numerically.

809. By reference to the COMPLETE SCHOOL ALGEBRA, page 238, it will be seen that the algebraic solution of a problem consists of two parts: 1st. The *Statement*, which is the expressing by one or more equations of the conditions of the problem, *i. e.*, the relations between the known and unknown quantities (parts of the figure) to be compared; and 2d. The *Solution* of these equations, so as to find the values of the unknown quantities in known ones.

810. In applying the equation for the solution of such problems as are now proposed, we have to depend upon our previously acquired knowledge of the properties of geometrical figures for the relations between the known and unknown quantities, which will enable us to form the necessary equations, *i. e.*, to make the *State-*

ment. The resolution of the equations thus arising is effected in the ordinary ways. [See NOTE, page 239 of THE COMPLETE SCHOOL ALGEBRA.]

811. The details of this method will be most readily obtained from a careful study of examples.

EXAMPLES.

812. In a right angled triangle, given the hypotenuse and the sum of the other two sides, to find these sides separately.

SOLUTION.—Let ABC be a triangle, right angled at B. Let the *known* hypotenuse be h, the *unknown* base, y; the *unknown* altitude, x; and the *known* sum of the base and altitude, s.

We have here two unknown quantities, and hence must have two equations, in order to find their values. One of these equations is furnished directly by the statement of the problem, which says that the sum of the base and perpendicular is to be given. Hence—

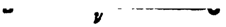


FIG. 429.

Equation 1 is $x + y = s$.

A second relation between x and y and the known quantity h is furnished by the relation given in PART II. (*346*). Whence—

Equation 2 is $x^2 + y^2 = h^2$.

Solving these equations we find—

$$y = \tfrac{1}{2}s \pm \tfrac{1}{2}\sqrt{2h^2 - s^2}, \text{ and } x = \tfrac{1}{2}s \mp \tfrac{1}{2}\sqrt{2h^2 - s^2}.$$

If $h = 10$ and $s = 14$, we find $x = 6$, and $y = 8$; or $x = 8$, and $y = 6$.

GEOMETRICAL SOLUTION.—It is exceedingly interesting and instructive to compare the algebraic solution of such problems with their geometrical solution, when the problem can be solved in both ways. The geometrical solution of this problem is as follows:

Take DC $= s$, the sum of the two sides, and make ODC $= 45°$. From C as a centre, with a radius h, the hypotenuse, describe an arc cutting DO, as in A and A′. Draw AC and the perpendicular AB, also A′C and the perpendicular A′B′. Both the triangles ABC and A′B′C fulfil the conditions. For AB = DB (?), whence AB + BC $= s$, and AC $= h$, by construction. So, also, A′B′ = DB′ (?), whence A′B′ + B′C $= s$, and A′C $= h$, by construction.

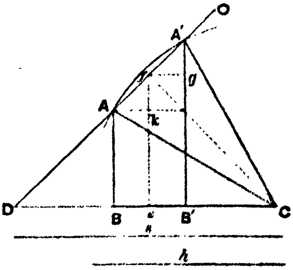


FIG. 430.

COMPARISONS OF THESE SOLUTIONS.—1st. We find in the *algebraic* solution, that, in general, y may have two values—viz., $\frac{1}{2}s + \frac{1}{2}\sqrt{2h^2 - s^2}$, and $\frac{1}{2}s - \frac{1}{2}\sqrt{2h^2 - s^2}$; and that when $y = \frac{1}{2}s + \frac{1}{2}\sqrt{2h^2 - s^2}$, $x = \frac{1}{2}s - \frac{1}{2}\sqrt{2h^2 - s^2}$; but, when $y = \frac{1}{2}s - \frac{1}{2}\sqrt{2h^2 - s^2}$, $x = \frac{1}{2}s + \frac{1}{2}\sqrt{2h^2 - s^2}$. Correspondingly, we find in the *geometrical* solution that the base (y) may have two values—viz., BC, and B′C; and that when the base is BC, the altitude (x) is AB; but, when the base is B′C, the altitude is A′B′.

2d. From the algebraic solution, we observe that the base $y = \frac{1}{2}s \pm \frac{1}{2}\sqrt{2h^2 - s^2}$, may be considered as made up of two parts—viz., a rational part, $\frac{1}{2}s$, and a radical part, $\frac{1}{2}\sqrt{2h^2 - s^2}$; and that the altitude, $x = \frac{1}{2}s \mp \frac{1}{2}\sqrt{2h^2 - s^2}$, is made up of the *same* parts, only observing that, if the base is considered as the *sum* of these parts—viz., $\frac{1}{2}s + \frac{1}{2}\sqrt{2h^2 - s^2}$, the altitude is their *difference*—viz., $\frac{1}{2}s - \frac{1}{2}\sqrt{2h^2 - s^2}$. If, however, the base is $\frac{1}{2}s - \frac{1}{2}\sqrt{2h^2 - s^2}$, the altitude is $\frac{1}{2}s + \frac{1}{2}\sqrt{2h^2 - s^2}$. Now, we can discover exactly the same things *geometrically*, and can show exactly what is the geometrical meaning of each of the parts of the values of y and x. To do this, draw C*f* * bisecting AA′; let fall the perpendicular *fe*, and draw A*k* and *fg* parallel to DC. C*f* is perpendicular to DO (?), and hence equal to D*f* (?). Also, D*e* = *e*C = *fe* = $\frac{1}{2}s$ (?). From the right angled isosceles triangle D*f*C, $fC = \frac{1}{\sqrt{2}} DC = \frac{s}{\sqrt{2}}$ (?). Hence, from A*f*C, $Af =$

$$AC^2 - \overline{fC}^2 = \sqrt{h^2 - \tfrac{1}{2}s^2} = \frac{1}{\sqrt{2}}\sqrt{2h^2 - s^2}.$$

Again, from the right angled isosceles triangle A*kf*, we have $Ak = \frac{1}{\sqrt{2}} Af\,(?) = \frac{1}{2}\sqrt{2h^2 - s^2}$. But $Ak = fk = fg = A'g = Be = eB'$. Hence we see that the rational part of the value of y ($\frac{1}{2}s$) is *e*C, and that the radical part ($\frac{1}{2}\sqrt{2h^2 - s^2}$) is B*e*, or *e*B′. In the triangle ABC the *sum* of these parts is the base; and in the triangle A′B′C, their *difference* is the base. In like manner *fe* represents the rational part of the value of x, and $fk = A'g$, the radical part.

3d. From the algebraic solution we see that if $s^2 = 2h^2$, $y = \frac{1}{2}s$, and $x = \frac{1}{2}s$. The same thing is seen in the geometrical solution, for if $s^2 = 2h^2$, $h = \frac{1}{\sqrt{2}}s$, or *f*C; whence the arc struck from C as a centre, with h as a radius, would be tangent to DO, instead of intersecting it in *two* points. Again, if $s^2 > 2h^2$, the quantity under the radical sign is negative, and the radical becomes *imaginary*. This means, that *no triangle* can be formed under these circumstances. This case appears in the geometrical solution also, for then $h < \frac{1}{\sqrt{2}}s$, or less than *f*C, and consequently the arc struck from C as a centre, with radius h, will not touch DO, and we get no triangle.

---

* This part of the construction should not be allowed on the figure till it is wanted—*i.e.*, till this stage of the discussion.

813. Sch.—This problem is discussed thus at length as an illustration of what *may* be done by such methods. Of course, all problems are not equally fruitful; but the student should not rest satisfied with a mere determination of the values of the unknown parts in known terms, when anything farther is revealed either by the process or result of the algebraic solution. Especially should he desire to become expert in seeing what geometrical relations are indicated by the *form* of the answer obtained.

---

814. Given the lengths of the medial lines from the acute angles of a right angled triangle, to determine the triangle, *i. e.*, to find the base and perpendicular.

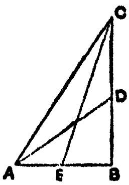


Fig. 431.

Sug's.—Let AD $= a$, CE $= b$, AB $= 2x$, and CB $= 2y$; then $4x^2 + y^2 = a^2$, and $4y^2 + x^2 = b^2$ (?). $\therefore 2x =$ AB $= 2\sqrt{\frac{4a^2 - b^2}{15}}$, and $2y =$ CB $= 2\sqrt{\frac{4b^2 - a^2}{15}}$.

The *form* of these results indicates that CB sustains the same relation to CE and AD that AB does to AD and CE—a fact which is evident from the nature of the case.

Again, if $4a^2 < b^2$, $2x$ is imaginary; and if $4b^2 < a^2$, $2y$ is imaginary. In either case the triangle cannot exist. So also if $4a^2 = b^2, 2x = 0$; and if $4b^2 = a^2, 2y = 0$, and there can be no triangle. This may be seen from the figure by conceiving AB, for example, to diminish. As A approaches B, AD approaches equality with DB, and CE with CB. Hence the *limit* is AD $= \frac{1}{2}$CE.

Thus we see *that either medial line must be more than half the other*,—a proposition which is proved by this solution.

---

815. The hypotenuse and radius of the inscribed circle of a right angled triangle being given, to determine the triangle.

Results.—Calling the hypotenuse h, the radius r, the base x, and the perpendicular y, we have, $x = \frac{2r + h \pm \sqrt{h^2 - 4hr - 4r^2}}{2}$, and

$$y = \frac{2r + h \mp \sqrt{h^2 - 4hr - 4r^2}}{2}.$$

The results being the same in other respects, the double sign before the radical indicates that the base and perpendicular are interchangeable—a fact which is evident from the nature of the case.

If the radical is 0, *i. e.*, if $h^2 - 4hr - 4r^2 = 0$, $x = r + \frac{1}{2}h$, and $y = r + \frac{1}{2}h$, and the base and perpendicular are equal. Let the student show the same thing geometrically (from a figure).

Also, if $h^2 - 4hr - 4r^2 = 0$, $h = 2r\,(1 \pm \sqrt{2})$. In this result the negative sign is to be rejected, since it would make h negative, as $\sqrt{2} > 1$.

The value $h = 2r\,(1 + \sqrt{2})$ is readily seen from the figure when AB = CB. Thus AC = 2DB = 2 (DO + OB) = $2\,(r + r\sqrt{2}) = 2r\,(1 + \sqrt{2})$ (?).

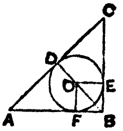


FIG. 432.

---

816. A tree of known height standing perpendicular on a horizontal plane, breaks so that its top strikes the ground at a given distance from the foot, while the other end hangs on the stump. How high is the stump? That is, given the base and the sum of the perpendicular and hypotenuse of a right angled triangle, to determine the perpendicular.

Result.—Let a be the height of the tree, b the distance from the foot to the point where the top strikes, and x the height of the stump; then $x = \frac{a^2 - b^2}{2a}$.

Since $\frac{a^2 - b^2}{2a} = \frac{1}{2}a - \frac{b^2}{2a}$, $\frac{b^2}{2a}$ is the distance below the middle, at which the tree breaks.

---

817. In a rectangle, knowing the diagonal and perimeter, to find the sides.

---

818. Knowing the base, b, and altitude, a, of any triangle, to find the side of the inscribed square, x.

Result, $x = \frac{ab}{a + b}$.

---

819. In an equilateral triangle, given the lengths, a, b, c, of the three perpendiculars from a point within upon the sides, to determine the sides.

SUG'S.—Find an expression for the altitude in terms of the sides; and then get two expressions for the area of the whole triangle. Equate these.

Result, each side $= \frac{2\,(a + b + c)}{\sqrt{3}}$.

---

820. In a right angled triangle whose hypotenuse is h, and difference between the base and perpendicular d, to find these sides.

Results, $x = \frac{-\,d \pm \sqrt{2h^2 - d^2}}{2}$, $x + d = \frac{d \pm \sqrt{2h^2 - d^2}}{2}$.

QUERIES.—Why must the minus sign of the radical be omitted in the geometrical interpretation of these results? What is the least possible value of d? What are the sides of the triangle for these values? What is the superior limit of the value of d? What do the sides become at this limit?

---

821. In an equilateral triangle given the lines a, b, c, drawn to its three vertices from a point within or without, to find the sides.

Result.—Each side =

$$\left\{\frac{a^2 + b^2 + c^2 \pm \sqrt{6\,(a^2b^2 + b^2c^2 + c^2a^2) - 3\,(a^4 + b^4 + c^4)}}{2}\right\}^{\frac{1}{2}}$$

The radical is + when the point is within, and − when it is without.

---

822. The perimeter of a right angled triangle and the perpendicular from the right angle upon the hypotenuse being given, to determine the triangle.

SUG'S.—Let s be the perimeter, p the perpendicular upon the hypotenuse, and $x + y$, $x - y$ the two sides about the right angle. Then the hypotenuse $= s - 2x$, and we readily form the two equations $p\ (s - 2x) = x^2 - y^2$, and $(x + y)^2 + (x - y)^2 = (s - 2x)^2$ (?). Hence $x = \frac{s(s + 2p)}{4(s + p)}$, and this value substituted in either equation will give y.

---

823. The base of a plane triangle is b and its altitude a, required the distance from the vertex at which a parallel to the base must cut the altitude in order to bisect the triangle.

Result, $\dfrac{a}{\sqrt{2}}$.

QUERY.—What does the fact that b does not appear in the result show?

---

824. Having given the area of a rectangle inscribed in a triangle, can the triangle be determined? Can it, if the rectangle is a square? If the rectangle is a square and the triangle right angled? If the rectangle is a square and the triangle equilateral?

---

825. The sides of a triangle being a, b, c, to find the perpendicular upon c from the opposite angle.

$$\textit{Result, } p = \frac{1}{2c}\sqrt{2c^2\,(a^2 + b^2) + 2a^2b^2 - a^4 - b^4 - c^4}.$$

SUG'S.—Observe that a and b are similarly involved in the result, but c is differently involved from either. This is evidently as it should be, since a and b are the sides about the angle from which p is let fall; and are thus similarly related to p. But c, the side on which p falls, is differently related to p from

either of the others. The student should be able to write the value of the perpendiculars upon each of the other sides, from this one. Thus, that on a is

$$\frac{1}{2a}\sqrt{2a^2(c^2+b^2)+2c^2b^2-a^4-b^4-c^4}.$$

826. The sides of a triangle are a, b, c, to find the side of an inscribed square one of whose sides falls in c.

SUG'S.—The altitude may be found from the preceding, hence may be assumed as known. Call it p. Then the side of the required square is $\frac{cp}{c+p}$. What is the side of the square standing on a? On b?

QUERY.—Will the square be the same on whichever side it stands? Observe that though the values here found are apparently different, they *may* not be so really, since p is different in each case. But let the student decide.

827. Having the area of a rectangle inscribed in a given triangle and standing on a specified side, to determine the sides of the rectangle.

Result, b being the base on which the rectangle stands, p the altitude from this base, and s the given area, we have for the sides

$$x=\frac{b}{2}\pm\sqrt{\frac{b^2}{4}-\frac{sb}{p}},\ \text{and}\ y=\frac{p}{2}\mp\sqrt{\frac{p^2}{4}-\frac{sp}{b}}.$$

SUG'S.—The ± and ∓ signs indicate that, in general, there can be two equal rectangles inscribed standing on the same base. The student will do well to illustrate it with definite numerical values, as $p = 10$, $b = 6$, $s = 10$.

Again, $\frac{b^2}{4}$ must be greater than $\frac{sb}{p}$, and $\frac{p^2}{4} > \frac{sp}{b}$, *i. e.*, s must be less than $\frac{1}{4}pb$. That is, the greatest rectangle is half the area of the triangle, since $\frac{1}{2}pb$ is the area of the triangle.

◆

828. The Algebraic solution of a problem often enables us to effect a geometrical construction. We will give a few examples.

Through a given point within a circle, to draw a chord of a given length.

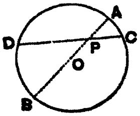


FIG. 433.

SOLUTION.—Let s be the length of the required chord, and P the given point. Since P is a known point, call AP $= a$, PB $= b$, AB being the diameter through P. Let CD represent the required chord, and calling CP, x, PD $= s - x$. Then $sx - x^2 = ab$; whence $x = \frac{1}{2}s \pm \sqrt{\frac{1}{4}s - ab}$.

To effect the geometrical construction, let s be the length of the given chord, and P the point in the given circle. Draw the diameter through P, and erect PE perpendicular to it. Make EH $= \frac{1}{2}s$; then since $\overline{PE}^2 = ab$, PH $= \sqrt{\frac{1}{4}s^2 - ab}$. Now take HI $= \frac{1}{2}s$, and from P as a centre, with a radius PI $= \frac{1}{2}s + \sqrt{\frac{1}{4}s^2 - ab}$, strike the arc DI intersecting the circumference. DPC is the chord required.

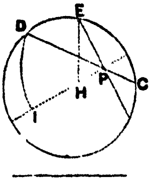


FIG. 434.

From the radical $\sqrt{\frac{1}{4}s^2 - ab}$ we see that, if $ab > \frac{1}{4}s^2$, x is imaginary, as we say in algebra. In such a case the problem is geometrically impossible, as will appear from the construction, for then PE is greater than EH, which makes HP, the representative of $\sqrt{\frac{1}{4}s^2 - ab}$, impossible. If $\frac{1}{4}s^2 = ab$, x has but one value, and the segments are equal.

829. To find a point in a tangent to a circle from which, if a secant be drawn to the extremity of the diameter passing through the point of tangency, the external segment shall have a given length.

SOLUTION.—Let AB $= d$ be the diameter of the given circle, DX $= a$ the external segment of the required secant, and the whole secant BX $= x$. Then

$$x^2 - ax = d^2, \text{ and } x = \tfrac{1}{2}a \pm \sqrt{d^2 + \tfrac{1}{4}a^2}.$$

To effect the geometrical construction, construct the radical by taking AC $= \frac{1}{2}a$; whence BC $= \sqrt{d^2 + \frac{1}{4}a^2}$. Now make CY $= \frac{1}{2}a$, and with B as a centre, and BY as a radius, strike an arc cutting the tangent, as in X. Then is

$$\text{BX} = x = \tfrac{1}{2}a + \sqrt{d^2 + \tfrac{1}{4}a^2}.$$

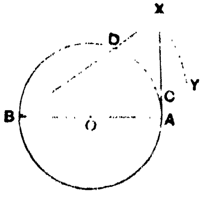


FIG. 435.

The negative value of the radical is inapplicable in this elementary, geometrical sense, since as $\sqrt{d^2 + \frac{1}{4}a^2} > \frac{1}{2}a$, this would make x a negative quantity. Again we see that no real value of a can render x imaginary.

We can observe the same things from the geometrical construction. Thus, if the negative value of the radical were taken, x would be *numerically* less than BC, by $\frac{1}{2}a$, or AC. But BC − AC < BA. Hence an arc struck from B with the required radius would not cut the tangent. We see also that a may have any value between 0 and ∞.

830. Given the hypotenuse and area of a right angled triangle, to construct the triangle.

SUG'S.—Let h be the hypotenuse, s^2 the area, and x the perpendicular from the right angle upon the hypotenuse. Then $hx = 2s^2$, or $\frac{1}{2}h : s :: s : x$, and $h : 2s :: 2s : 2x$.

The figure will suggest the construction.

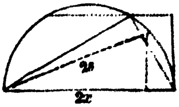


FIG. 436.

831. Through a point between two lines which intersect, to draw a line which shall cut off a triangle of given area.

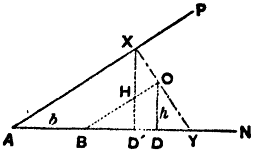


FIG. 437.

SUG'S.—Let AY $= x$, and the required area $= s^2$. We have $h : \mathrm{H} :: x - b : x$. $\therefore \mathrm{H} = \dfrac{hx}{x-b}$. And $\mathrm{H}x = 2s^2$. $\therefore \mathrm{H} = \dfrac{2s^2}{x}$. Thus

$$x = \frac{s^2}{h} \pm \sqrt{\frac{s^2}{h}\left(\frac{s^2}{h} - 2b\right)}.$$

To construct this, find $c = \dfrac{s^2}{h}$, *i. e.*, construct a third proportional to h and s. Then construct $\sqrt{c(c-2b)}$, *i. e.*, find a mean proportional between c and $c - 2b$; let this be m. Whence $x = c \pm m$. In general, there may be two solutions, if any, since there are two values of x. [This should also be observed from the figure.] But if $2b > \dfrac{s^2}{h}$ there is no solution. If $\dfrac{s^2}{h} = 2b$, there is but one solution. In the latter case where is the given point O? What is the geometrical difficulty when $2b > \dfrac{s^2}{h}$? Can m be numerically greater than c?

832. To construct the four forms of the affected or complete quadratic equation, viz., (1.) $x^2 + px - q = 0$, (2.) $x^2 - px - q = 0$, (3.) $x^2 - px + q = 0$, (4.) $x^2 + px + q = 0$, without solving the equations.

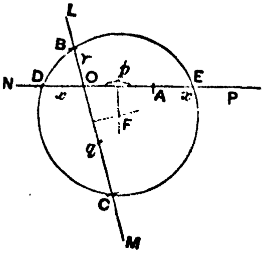


FIG. 488.

FIRST FORM. $x^2 + px - q = 0$.—Draw any two lines as LM, NP, intersecting in some point O. Resolve q of the equation into two factors, as r and q', so that we have $x^2 + px - r \times q' = 0$. Take OA $= p$, OB $= r$, OC $= q'$. Bisect CB and AO by perpendiculars, and from their intersection F as a centre, with a radius FB, draw a circle. Then DO, or AE, is x, the positive root. For $x(x + p) = rq'$, or $x^2 + px - rq' = 0$. The negative root is OE. Thus, let OE $= (-x)$. Then DO $=$ AE $= (-x - p)$. Hence $(-x)(-x - p) = x^2 + px = rq'$, or $x^2 + px - rq' = 0$.

This construction is evidently always possible irrespective of the relative magnitudes of p, r, q'; a fact which agrees with the statement in algebra that this form always has *real roots*.

SECOND FORM. $x^2 - px - rq' = 0$.—The construction is the same as for the first form; only, in this case OE is the positive, and DO the negative root. Thus for OE $= x$ (positive), we have DO $\times$ OE $= (x - p)\,x = rq'$, or $x^2 - px -$

$rq' = 0$. For $\mathsf{DO} = (-x)$, we have $\mathsf{DO} \times \mathsf{OE} = \mathsf{DO}(\mathsf{OA} + \mathsf{AE}) = \mathsf{DO}(\mathsf{OA} + \mathsf{DO}) = (-x)(p - x) = rq'$, or $x^2 - px - rq' = 0$.

Observe that in the first case the negative root is numerically greater than the positive; while it is the reverse in this form. This agrees with the conclusions of algebra (See COMPLETE SCHOOL ALGEBRA, *104*).

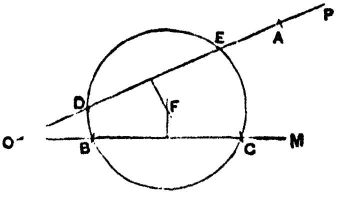


FIG. 439.

THIRD FORM. $x^2 - px + rq' = 0$.—Draw any two lines, as **OM**, **OP**, meeting at **O**. Take $\mathsf{OA} = p$, $\mathsf{OB} = r$ or q', and $\mathsf{OC} = q'$ or r. Erect perpendiculars at the middle points of **OA**, and **BC**; and from their intersection **F** as a centre, with a radius **FB**, strike a circumference. Then **OE** and **OD** are the values of x. For $\mathsf{OE} = x$, $\mathsf{OE} \times \mathsf{OD} = \mathsf{OE} \times \mathsf{EA} = \mathsf{OE}(\mathsf{OA} - \mathsf{OE}) = x(p - x) = rq'$, or $x^2 - px + rq' = 0$. For $\mathsf{OD} = x$, $\mathsf{OD} \times \mathsf{OE} = \mathsf{OD}(\mathsf{OA} - \mathsf{AE}) = \mathsf{OD}(\mathsf{OA} - \mathsf{OD}) = x(p - x) = rq'$, or $x^2 - px + rq' = 0$.

Observe that the former value of x is greater than the latter, but that neither is negative.

So also, we may readily see that the roots may become equal, and also, imaginary. Thus if the circle were tangent to **OA**, the roots would be equal, and if it did not touch **OA** they would *both* be imaginary. (See Algebra, as above.)

FOURTH FORM. $x^2 + px + rq' = 0$.—The construction is the same as the last, only both values of x are negative. Thus, $(-x)[p - (-x)] = (-x)(p + x) = rq'$, $-px - x^2 - rq' = 0$, or $x^2 + px + rq' = 0$.

SCH.—Thus we see that we can construct any equation of the second degree containing but one unknown quantity, which has real roots. Hence, if the algebraic solution of a geometrical problem requires only the resolution of such an equation, the algebraic solution will lead to the geometrical construction.

333. We have now given sufficient illustrations of this most interesting and important subject, so that the student should have caught the spirit of this method of using algebra to subserve the purposes of geometrical investigation. We shall simply append a list of problems, upon which the student can put in exercise both his algebraic and geometric knowledge. But we cannot refrain from repeating the advice, that the learner should not rest satisfied with the mere algebraic resolution of the problem. He should be ambitious to trace, *as fully as possible,* the wonderful relations which exist between the abstract operations of algebra, and the more concrete relations of geometry.

EXAMPLES.

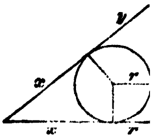


FIG. 440.

834. Given the perimeter of a right angled triangle and the radius of the inscribed circle, to determine the triangle.

835. Given the hypotenuse of a right angled triangle and the side of the inscribed square, to determine the triangle.

---

836. In a right angled triangle, given the radius of the inscribed circle, and the side of the inscribed square, the right angle of the triangle constituting one angle of the square, to determine the triangle.

SUG'S.—Letting x and y be the sides, z the hypotenuse, r the radius of the inscribed circle, and s the side of the inscribed square, we have $s = \frac{xy}{x + y}$, $xy = r\,(x + y + z)$, and $x + y = z + 2r$. Whence $z = 2r\left(\frac{s - r}{2r - s}\right)$, etc

---

837. In any triangle whose sides are a, b, c, to find the radius of the inscribed circle.

838. Show that the area of a regular dodecagon inscribed in a circle whose radius is 1, is 3.

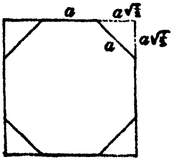


FIG. 441.

839. Find the area of a regular octagon whose side is a.

Result, $2\,(\sqrt{2} + 1)\,a^2$.

840. Find the radii of three equal circles described in a given circle, tangent to the given circle and to each other.

---

841. The space between three equal circles tangent to each other is a; what is the radius?

---

842. In a triangle, given the ratio of two sides, and the segments of the third side made by a perpendicular let fall from the angle opposite.

843. In a triangle, given the base and altitude, and the ratio of the other sides, to determine the triangle.

---

844. Given the base, the medial line, and the sum of the other sides of a triangle, to determine the triangle.

---

845. To determine a right angled triangle, knowing the perimeter and area.

SUG'S. $x^2 + y^2 = z^2$, $x + y + z = 2p$, and $xy = 2s^2$, give $y + x = 2p - z$, $x^2 + 2xy + y^2 = 4p^2 - 4pz + z^2$, $z^2 + 4s^2 = 4p^2 - 4pz + z^2$; whence $z = \frac{p^2 - s^2}{p}$. Now use $y + x = 2p - z = \frac{p^2 + s^2}{p}$, and $xy = 2s^2$.

---

846. To determine a right angled triangle, knowing the perimeter, and the sum of the hypotenuse, and the perpendicular upon the hypotenuse from the right angle.

SUG'S. $x^2 + y^2 = z^2$, $x + y + z = 2p$, $z + u = a$, $xy = zu$. Then $x^2 + 2xy + y^2 = 4p^2 - 4pz + z^2$; whence $2xy = 4p^2 - 4pz$, and hence $2z(a - z) = 4p - 4pz$, etc.

---

847. The volume, the altitude, and a side of one of the bases of the frustum of a square pyramid being known, to determine a side of the other base.

---

848. To determine a right angled triangle, knowing the perimeter, and the perpendicular let fall from the right angle upon the hypotenuse.

---

849. To determine a triangle, knowing the base, the altitude, and the difference of the other sides.

---

850. To determine a triangle, knowing the base, the altitude, and the rectangle of the other sides.

---

851. To determine a right angled triangle, knowing the hypotenuse and the difference between the lines drawn from the acute angles to the centre of the inscribed circle.

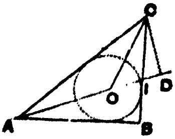


FIG. 442.

SUG'S.—Let fall **CD** a perpendicular upon **AO** produced. Now, since the the angles **BAC** and **ACB** are bisected, and **COD** = **OAC** + **OCA**, and **ICD** = **IAB**, they being complements of the equal angles **CID**, **IAB**, we have, **COD** = **OCD**, and **CD** = **OD** = $\sqrt{\frac{1}{2}}$ **CO**. Hence, putting **AC** = h, **CO** = x, and **AO** = $x + d$, we have

$$(x + d + \sqrt{\tfrac{1}{2}}\, x)^2 + (\sqrt{\tfrac{1}{2}}\, x)^2 = h^2.$$ From this x is readily found. The student should then be able to complete the solution.

852. Given two sides of a triangle and the bisector of their included angle, to determine the triangle.

---

853. Given the three medial lines, to determine a triangle.

---

854. Given the three sides of a triangle, to determine the radius of the circumscribed circle.

855. Four equal balls whose radius is r are placed on a plane so that each is tangent to the other three, thus forming a pyramid; what is its altitude?

856. Given the base of a triangle, the bisector of the opposite angle, and the radius of the circumscribing circle, to determine the triangle.

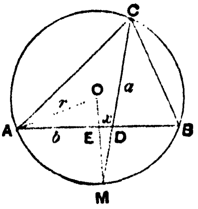


FIG. 448.

SUG'S.—First to find ED $= x$. Since EM $= r - \sqrt{r^2 - b^2}$, it may be considered known and put equal to c. We then have DM $= \sqrt{c^2 + x^2}$; and also, DM $\times\ a =$ AD $\times$ DB $= b^2 - x^2$, or DM $= \dfrac{b^2 - x^2}{a}$.

Whence $\sqrt{c^2 + x^2} = \dfrac{b^2 - x^2}{a}$, and x is readily found. Calling ED $= s$, the student will have no difficulty in proceeding with the solution.

CHAPTER II.

INTRODUCTION TO MODERN GEOMETRY.[*]

SECTION I.

OF LOCI.

857. The term ***Locus***[†], as used in geometry, is nearly synonymous with *geometrical figure,* yet having a latitude in its use which the other does not possess. *The locus of a point* is the line (geo-

---

[*] With strict propriety only the latter sections of this chapter belong to the *Modern Geometry,* technically so called. But, as the entire chapter is composed of matter which has not hitherto found place in our common text-books, and the relative importance of which is becoming more fully appreciated in modern times, the author has ventured to embrace the whole under this title.

[†] The word *Locus* is the l

metrical figure) generated by the motion of the point according to some given law.

In the same manner, a surface is conceived as the locus of a line moving in some determinate manner.

ILL'S.—The locus of a point in a plane, which point is always equidistant from the extremities of a given right line, is a straight line perpendicular to the given line at its middle point. Thus, suppose AB a fixed line, and *the locus of a point* equidistant from its extremities is required; that point may be anywhere in a perpendicular to AB at its middle point, *and cannot be anywhere else* in this plane.

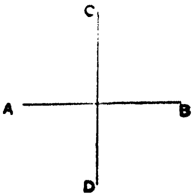


FIG. 444.

This perpendicular is the locus (place) of a point subject to the given law.

Again, a boy on the green is required to keep at just 20 feet from a certain stake; where may he be found? *i. e.*, what is his locus (place)? Evidently, the circumference of a circle whose radius is 20 feet. Thus, the locus of a point in a plane, equidistant from a given point, is the circumference of a circle. This is the *place* of such a point.

What is the locus in space of a point equidistant from a given point?

What is the locus of a point in space equidistant from the extremities of a given line? A plane.

What is the locus of a line moving so that each point in it traces a right line? In general, a plane; if it move in the direction of its length, a straight line.

What is the locus of a right line parallel to and equidistant from a given line?

What is the locus of a right line intersecting a given line at a constant angle? * A conical surface of revolution.

What is the locus of a semicircle revolving on its diameter?

PROPOSITIONS AND PROBLEMS IN DETERMINING PLANE LOCI.

[NOTE.—The student should be required to give every demonstration *in form*, and in detail. Frequent exercise in writing out demonstrations, is almost the only method of securing a good, independent style in demonstration.]

858. Theo.—*The locus of a point in a plane, equidistant from the extremities of a given line, is a perpendicular to that line at its middle point.*

SUG.—To prove this we have simply to show that every point in such a perpendicular is equidistant from the extremities of the given line, and that no other point has this property (PART II., *120*).

---

* That is, an angle which remains of the same size.

859. Prob.—*Find the locus of a point at any constant distance* m *from a straight line.* Of what proposition in PART II. is this the converse?

SUG'S.—To prove the proposition which the answer to this question asserts, it will be necessary to show that every point in the affirmed locus is at the same distance from the given line and that *no other* point is at that distance. We affirm that *the locus is two right lines parallel to the given line and at a distance* m *therefrom.* The formal demonstration is as follows: Let AB be the given line, and OE, OE′, perpendiculars thereto, each equal to *m*. Through E and E′ draw CD and C′D′ parallel to AB; then is CD, C′D′, the locus required.* For, by Part II. (*156*), *every* point in CD, C′D′, is at the distance *m* from AB; and we may readily show that any other point, as P or P′, is at a distance greater or less than *m* from AB. Hence CD, C′D′, is the locus required.

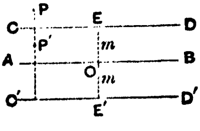


FIG. 445.

860. Theo.—*In a circle, the locus of the centre of a chord parallel to a given line is a diameter.*

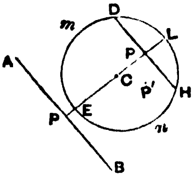


FIG. 446.

DEM.—Let *mn* be any circle, and AB a given line. Then is the locus of the centre of a chord parallel to AB, a diameter of the circle.

For, let DH be *any* chord parallel to AB. Through the centre of the circle C, and P, the middle point of DH, draw EL. Now EL is *perpendicular* to DH (?), and consequently to AB (?). Then will EL be perpendicular to *any* and every chord parallel to DH (?), and hence will bisect such chord (?). Therefore the locus of the centre of a chord parallel to AB is a diameter.

Again, any point in the circle and out of the line EL is not the middle point of chord parallel to AB. Thus, letting P′ be such a point, draw a chord through P′ parallel to AB. As there can be but one such chord (?), and as EL bisects it (?), P′ is without the diameter (?).

861. Theo.—*The locus of the centre of a circumference passing through two given points is a straight line.*

SUG.—Consult PART II. (*159, 163, 197*). The student should put the argument in form.

862. Theo.—*The locus of the centre of a circle which is tan-*

---

* It is important that the student think of these two lines as *one* locus, or as *parts* of *one and the same* locus, if this will aid the conception. A locus may consist of any number of detached parts; all that is necessary being that the given conditions be fulfilled. In this respect the word *locus* has a more enlarged meaning than the term *geometrical figure.*

gent to a given circle at a given point, is a straight line passing through the centre of the given circle.

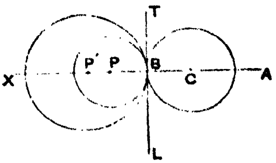


FIG. 447.

DEM.—Let **C** be the centre of the given circle, and **B** the point in the circumference to which the circle * shall be tangent, the locus of whose centre is required. Through **B** draw **TL** tangent to the given circle. Now, a circle passing through **B**, and tangent to the given circle, will have **TL** for its tangent (?), and as a radius is perpendicular to a tangent at its extremity, and only one perpendicular can be drawn to **TL** through **B**, the centre of a circle tangent to the given circle at **B** must be in this straight line. Moreover, as the given circle is tangent to the right line **TL** at **B**, its centre is in the perpendicular **AX**. Hence **AX** is the locus required.

863. Theo.—*The locus of the centre of a circle of given radius* R, *and tangent to a given straight line, is two parallels to this line at a distance* R *therefrom, on each side.* Give proof in form.

864. Prob.—*Find the locus of the centre of a circle of given radius* R, *whose circumference passes through a given point.* Give proof in form.

865. Theo.—*The locus of the centre of a line of constant length, having its extremities in two fixed lines which cut each other at right angles, is the circumference of a circle.*

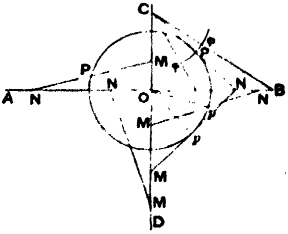


FIG. 448.

SUG'S.—Let **MN** be the length of the given line, and **CD**, and **AB**, the two lines intersecting at right angles, in which the extremities of **MN** are to remain. Now, in whatever position **MN** may be placed, its middle point, **P**, is at the same distance ($\frac{1}{2}$**MN**) from **O** (?). To show that any point not in this circumference, as φ, is not the middle point of a line equal to **MN** passing through it, and limited by the fixed lines, from φ as a centre, with a radius $\frac{1}{2}$**MN** cut **CD**, as in **C**; and from **C** as a centre with the same radius strike the arc φ**P**. If φ is without the circle, **CB** > **MN**, if within, less. Hence, the required locus is a circumference whose centre is **O**, and whose radius is $\frac{1}{2}$**MN**.

---

* Observe the form of expression. We say "the circle," and not "the circles," using the term in a *generic sense*, as including all which have the required property, *i. e.*, all which are tangent to the given circle at **B**.

***866.* Prob.**—*Find the locus of the centre of a chord of constant length, in a given circle.*

Sug.—We say, once again, *always give the proof in form.*

***867.* Prob.**—*Find the locus of the vertex of the right angle of a right angled triangle of a constant hypotenuse.*

***868.* Prob.**—*Find the locus of the middle point of the chord intercepted on a line through a given point, by a given circumference, when the given point is without the circumference, when it is in, and when it is within the circumference.*

***869.* Prob.**—*Find the locus of a point the sum of whose distances from two fixed intersecting lines is constant,* i. e., *is equal to a given line.*

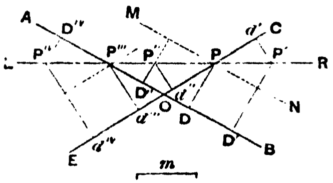


Fig. 449.

Solution.—Let AB and CE be the fixed lines, and m the constant distance. Draw MN parallel to AB, and at a distance m from it. Bisect the angle CPN. Then is LR (a part of) the locus required. For [the student will here show that the sum of the distances from any point in LR to AB and CE, as PD, $P''D'' + P''d''$, $P'D' + P'd'$, $P'''d'''$, $P^{iv}D^{iv} + P^{iv}d^{iv}$, is constant and equal to m], observe that when one of the perpendiculars measuring the distance from a point in the locus, changes from one side to the other of the line on which it is let fall, its *sign* changes. Thus $P''D''$, $P''d''$ being considered +, $P'd'$ and $P^{iv}D^{iv}$ are to be considered −. This is a general principal in mathematics. See Part II. (*215*), and foot note.

Finally, LR is only a part of the locus, since there is another line on the opposite side of AB, obtained by drawing the auxiliary MN on that side, which fulfills the same condition. The student should show what the result is when we draw the auxiliary parallel to CE, and on either side of it, also that any point not in one of these lines cannot fulfill the required condition. The complete locus is four indefinite right lines intersecting each other at right angles, so as to inclose a rectangle.

***870.* Prob.**—*Find the locus of a point such that the sum of the squares of its distances from two fixed points shall be equivalent to the square of the distance between the fixed points.*

***871.* Prob.**—*Find the locus of the intersection of two secants* *drawn through the extremities of a fixed diameter in a given circle, one of the secants being always perpendicular to a tangent to the circle at the point where the other cuts it.*

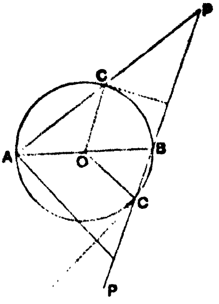


FIG. 450.

SUG'S. **P** being the point, show that **PB** = **AB**, for any position of **AP** and **BP**. Hence, any point in the circumference having **B** for its centre, and **AB** for its radius, fulfills the conditions. Show that any point out of the circumference does not fulfill the conditions.

872. Prob.—*Find the locus of the intersection of two lines drawn from the acute angles of a right angled triangle, through the points where the perpendicular to the hypotenuse cuts the opposite sides, or sides produced.*

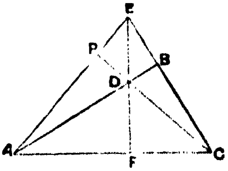


FIG. 451.

SUG'S.—The locus of **P** is required. Prove that **APC** is always a right angled triangle, wherever the perpendicular **EF** to the hypotenuse **AC** is drawn.

873. Prob.—*Find the locus of a point which divides a line drawn from a fixed point to a fixed line in a fixed ratio.*

SUG'S.—Most problems in finding loci such as are treated in Elementary Plane Geometry, viz., right lines and circles, are readily solved by constructing a few points according to the given conditions, whence we can determine by inspection whether the required locus is a right line or the circumference of a circle; and, having discovered this fact by inspection, it will remain to show *why it should be so.* Thus, in the present problem, **O** being the fixed point, and **AB** the fixed line, drawing a few lines, **OC**, according to the requirements, and dividing them in the same ratio (in the figure 3 : 2), we find a few points **P** in the locus. We then discover at once that the locus is a right line parallel to **AB**, and can easily see *why* it should be so.

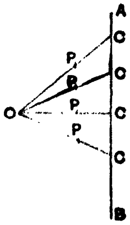
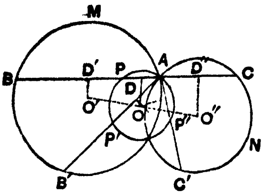


FIG. 452.

874. Prob.—*Two fixed circumferences intersect: to find the locus of the middle point of the line drawn through one of the points of intersection and terminated by its other intersections with the circumferences.*

SUG'S.—We will first give an example of the course which the mind of the student might take in his efforts to discover the solution. He would naturally draw two *unequal* * circles, as M and N, and, through one of the points of intersection, as A, draw BC, and bisect it at P. It is the locus of P that is desired. Now, suppose the line BC to revolve about A, B passing towards B′, and C towards A. It is the path of the middle point that he seeks. When C reaches A, the line becomes tangent to N, and P is the middle point of the chord AB′. In a similar manner, he sees that the middle point of the chord AC′, tangent to M at A, is also a point in the locus.



Again, he observes that as B moves towards M, and C towards N, P moves towards A, and when AC = AB, P is at A. It now appears probable that the locus of P is a circumference. Proceeding on this hypothesis, he reasons, that, if this is true, AP′ and AP″ are chords of the locus, and, bisecting them with perpendiculars, he will have the centre of the locus. Locating O thus, he observes that it *appears* to be in the line joining the centres O′ and O″, and about midway between them. This leads him to see, whether, by assuming the middle point of O′O″ as the centre of a circle, and OA as a radius, he can prove that any such line as BC drawn through A is bisected by this circumference, as at P. This he can readily prove by means of the perpendiculars OD, O′D′, and O″D″, which bisect the chords AP, AB, and AC. For, since these perpendiculars are parallel, and O′O = OO″, D′D = DD″; whence D′P = AD″, and, adding AP to each, D′A = PD″, or D′B = PD″. Adding to BD′, D′P, and D″C (= AD″ = D′P), there results BP = PC, and the hypothesis is true.

But, *in giving this problem as a recitation*, the student will proceed as follows: Letting M and N be the two fixed circumferences, intersecting at A, join their centres O′ and O″, and bisect O′O″ as at O. With O as a centre and OA as a radius, describe a circle. Then is this circumference the locus required. For, let BC be any secant line passing through A, we may show that P is the middle point of BC. [Having done this, as above, and shown that any point not in this circumference is not the middle of the secant line passing through A, his solution is complete.]

875. Prob.—*If the line* AB *is divided at* C, *find the locus of* P, *so that angle* APC = *angle* BPC.

SUG'S.—*In seeking for the solution*, the following would be a natural process. Drawing *any* line, as AB, in the lower part of the figure, taking C, any point in it, and *conceiving* BP and AP drawn so as to make *equal angles* with PC, we would naturally discover that, if a circle were circumscribed about BPA, PC produced would bisect the arc below AB. Thus we discover a ready method of locating P; *i. e.*, in the main figure, bisect AB by a perpendicular, as ED, and with any point on ED as a centre, pass a circumference through A and B. Through D and C draw a line, and P is a point in the locus (?). Any number of points can be found in this way; and, having found a few, as P, P', P'', P''', etc., the situation of these will suggest that, *probably*, the locus is the circumference of a circle whose centre is in AB produced. *If this should be the fact,* CP is a chord of that circle, and, erecting a perpendicular at the middle point of CP, its intersection with AB produced, as O, will be the centre of the locus (?). We will now endeavor to prove that *any* point in this circumference, as P', is so situated that BP'C = CP'A, and that no point out of this circumference has this property. We can readily show that the angle OPB = OCP − BPC = OCP − CPA (?). But PAC = OCP − CPA (?). ∴ Triangle OPB is similar to OPA (?), and OA : OP : : OP : OB, or OA : OC : : OC : OB. Now, for *any* point in this circumference, as P', we shall have OA : OP' : : OP' : OB, since OP' = OC, and OA, OC, and OB are constant. Hence, wherever P' is taken (in this circumference), the triangle OPB is similar to OPA, angle OP'B = P'AB, and BP'C = CP'A. Finally, that no point out of this circumference possesses this property is evident, since the distance of such a point from O would not equal OC, and the angle OP_1B (P_1 being such a point) would not equal P_1AB.

---

* *Equal* circles would probably have *special* relations.

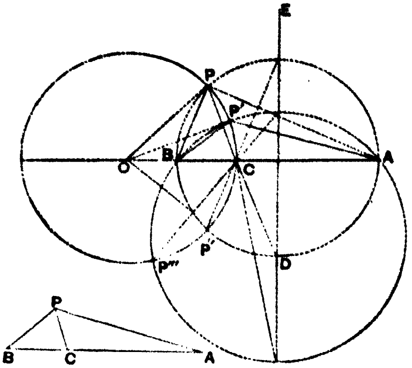


FIG. 454.

876. Prob.*—In a fixed circle, any two chords intersect at right angles in a fixed point; find the locus of the centre of the chord joining their extremities.* Give the proof.

877. Prob.*—Find the locus of the point in space equidistant from three given points.* Give the proof.

878. Prob.*—Find the locus of the point in space equidistant from two given points.* Give the proof.

879. Prob.*—Find the locus of the point in a plane such that the difference of the squares of the distances from it to two fixed points without the plane shall be constant.*

SUG'S.—Conceive the two points without the plane joined by a right line, and a perpendicular to this line drawn from either extremity of it; the point where this perpendicular pierces the plane is a point in the locus (?). The required locus is two parallel straight lines.

***880.* Prob.**—*Find the locus of the middle point of a straight line of constant length, whose extremities remain in two lines at right angles to each other, but which are not in the same plane.* Give the proof.

***881.* Prob.**—*Find the locus of the point equidistant from two fixed planes.* Give proof.

Sug's.—Consider, 1st, When the fixed planes are parallel; and 2d, when they intersect.

***882.* Prob.**—*What locus is the intersection of a plane and the surface of a sphere?* Give proof.

***883.* Prob.**—*What locus is the intersection of the surfaces of two given spheres?*

***884.* Prob.**—*Find the locus of the point in space such that the ratio of its distance from a given right line to its distance from a fixed point in that line is constant.*

SECTION II.

OF SYMMETRY.

885. Def.—Two points are said to be *symmetrical with respect to a third,* when the right line joining the two points is bisected by the point of reference, called the *Centre of Symmetry.*

886. Def.—Two loci, or two parts of the same locus, are *symmetrical with respect to a point,* when every point in one has its symmetrical point in the other.

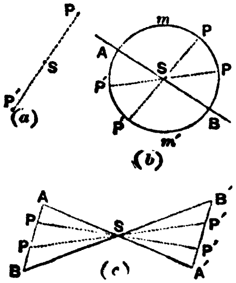


Fig. 455.

Ill's.—In (*a*) P' is symmetrical with P in respect to S, if S is the middle point of PP'. In (*b*) we observe that the semi-circumference A*m*'B is symmetrical with the semi-circumference A*m*B, in respect to the centre S; for any point P in the latter has a symmetrical point P' in the former. In (*c*) the triangle A'SB' is symmetrical with ASB in respect to S (?).

887. Theo.—*The symmetrical of a straight line, with respect to a point, is an equal straight line.*

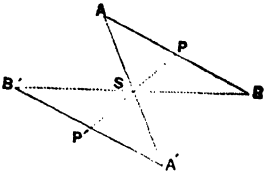


FIG. 456.

SUG'S.—We commence by assuming A'S = AS, and B'S = BS, and drawing B'A'. We then have to show that *any* point in AB, as P, has its symmetrical point P' in B'A'.

888. Theo.—*The symmetrical of an angle, with respect to a point, is an equal angle.*

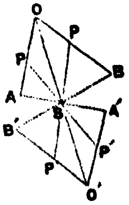


FIG. 457.

SUG'S.—To show the symmetrical of AOB, with respect to S, take A'S = AS, B'S = BS, O'S = OS, and draw O'B', O'A'. Then show that any point in OA has its symmetrical in O'A', and any point in OB has its symmetrical in O'B'. Hence, A'O'B' is the symmetrical of AOB, with respect to S.

Then apply AOB to A'O'B' and show that these symmetricals are equal.

889. Prob.—*Having given a polygon, to draw its symmetrical with respect to a given point.*

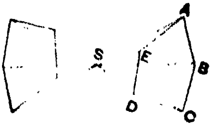


FIG. 458.

890. Theo.—*Any polygon is equal to its symmetrical with respect to a given point.*

SUG'S.—Proof by revolving the figure about S, keeping it in its own plane, each line, as AS, ES, etc., passing through 180°.

891. DEF.—The lines AS, ES, BS, etc., are *radii of symmetry*.

892. DEF.—Two points are *symmetrical with respect to a straight line* in the same plane, when the straight line which joins them is bisected at right angles by the line of reference, called the *Axis of Symmetry.*

893. DEF.—Two loci, or two parts of the same locus, are *symmetrical with respect to a straight line* when every point in the one has its symmetrical point in the other.

ILL'S.—Thus in (*a*) P and P' are symmetrical points with respect to the right line X'X, P's = Ps and X'X is perpendicular to PP'. So the part of (*b*) above X'X is symmetrical with the part below, *i. e.*, the curve is symmetrical with respect to X'X.

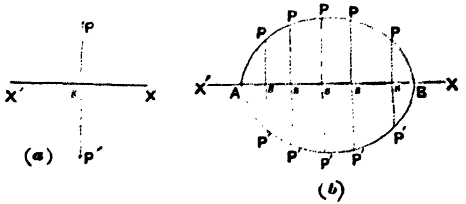


FIG. 459.

894. Theo.—*The symmetrical of a straight line, with respect to a rectilinear axis of symmetry, is an equal right line.*

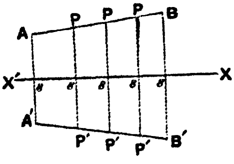


Fig. 460.

Dem.—Let AB be any straight line, and $X'X$ the axis. Let fall the perpendiculars As and Bs, and produce them till $A's = As$, and $B's = Bs$. Then is $AB = A'B'$, the symmetrical of AB. For, taking P, *any* point in AB, letting fall Ps, and producing it to P', the point P' is symmetrical with P; since revolving $sA'B's$ upon $X'X$, $A'B'$ will coincide with AB, and P' will fall at P. Hence $A'B' = AB$, and every point in AB has its symmetrical point in $A'B'$ (*893*).

895. Cor. 1.—*If two straight lines intersect, their symmetricals intersect, and the points of intersection are symmetrical.*

The student should show how this follows from the proposition.

896. Cor. 2.—*Two rectilinear symmetricals meet the axis in the same point, and make equal angles therewith.*

Student give proof.

897. Def.—A *Trapezoid* like $ABB'A'$, having its non-parallel sides equal, is called *Isosceles*.

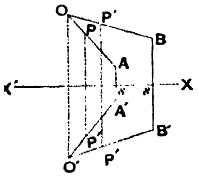


Fig. 461.

898. Theo.—*The symmetrical of an angle, with respect to a rectilinear axis of symmetry, is an equal angle.*

Sug's.—The student should be able to give the demonstration from the figure, in a manner altogether similar to the preceding; or, drawing AB and $A'B'$, he can base it upon the preceding.

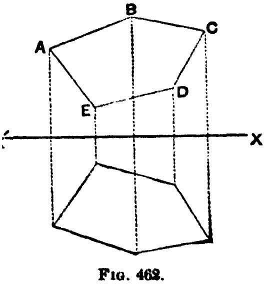


Fig. 462.

899. Prob.—*Having given a polygon, to draw its symmetrical with respect to a given axis.*

---

900. Theo.—*Any plane figure is equal to its symmetrical, with reference to a rectilinear axis.*

Proof by applying one to the other by revolution.

901. DEF.—Two loci in space, or two parts of the same locus (planes or solids), are symmetrical with respect to a point, when every point in one has a corresponding point in the other, such that the line joining them is bisected by the point called the centre of symmetry.

ILL'S.—Symmetrical triedrals (PART II., ***432***) afford an illustration of solids symmetrical with respect to a point—the vertex. The two hemispheres into which a great circle divides a sphere are symmetrical parts of the solid (sphere) with reference to the centre.

902. DEF.—Two points in space are symmetrical with respect to a plane called the *Plane of Symmetry*, when the line joining the points is perpendicular to the plane and bisected by it.

903. DEF.—Two loci in space (planes or solids), or two parts of the same locus, are symmetrical with respect to a plane when every point in one has its symmetrical point in the other.

904. DEF.—The corresponding (symmetrical) parts of symmetrical figures are called *Homologous* parts.

---

905. Theo.*—The symmetrical of a right line, with respect to a plane, is an equal right line.*

DEM.—Let AB be any right line, and MN the plane of symmetry. Let fall the perpendiculars B*b*, A*a*, upon the plane, produce them, making B′*b* = B*b*, A′*a* = A*a*, and join A′ and B′. Then A′B′ = AB, and is its symmetrical. For ABB′A′ being a plane figure (?) and *ab*, the intersection of this plane with MN, being a right line bisecting AA′ and BB′ at right angles (?), we may revolve *ab*B′A′ upon *ab* and bring A′B′ into coincidence with AB. Hence A′B′ = AB. Again, P being any point in AB, draw PP′ perpendicular to *ab*, and upon revolution P′ will fall in P, and P′*s* = P*s*. Hence, every point in AB has its symmetrical in A′B′, and the latter line is the symmetrical of the former.

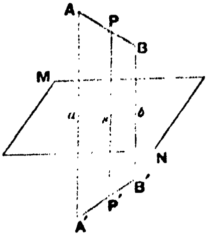


FIG. 463.

906. COR. 1.—*A right line and its symmetrical, with respect to a plane, pierce the plane at the same point.*

Student give proof.

---

907. Theo.*—The symmetrical of a plane angle, with respect to a plane, is an equal plane angle.*

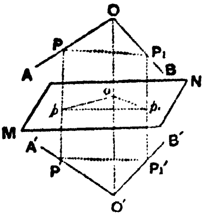


Fig. 464.

Dem.—Let **AOB** be any plane angle, and **MN** the plane of symmetry. Let **P** be any point in **AO**, and P_1 any point in **OB**. Let **O'** be the symmetrical of **O**, **P'** of **P**, and P_1' of P_1; then is **A'O'** the symmetrical of **AO**, **O'B'** of **OB**, and angle **A'O'B'** of **AOB**. Now by the preceding proposition the two triangles POP_1, and $P'O'P_1'$, are mutually equilateral, whence **AOB** = its symmetrical **A'O'B'**.

Query.—When will the triangle *pop'* exist, and when not?

908. Theo.*—Any plane polygon has for its symmetrical, with reference to a plane, an equal plane polygon.*

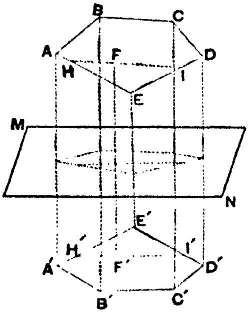


Fig. 465.

Sug's.—**ABCDE** being any plane polygon, and **MN** the plane of symmetry, by constructing **A',B',C',D',E'** symmetrical with **A, B, C, D, E,** we have by the preceding propositions **A'B'C'D'E'** equilateral and equiangular with **ABCDE**; whence it only remains to show that **A'B'C'D'E'** is a plane (not a warped) surface. Let **F** be any point in the angle **AED**, draw **HI**, and let **H'** and **I'** be the symmetricals of **H** and **I** (*895*). Draw **H'I'**. Then is the symmetrical of **F** in **H'I'** (?), as at **F'**. Now, every point in **HF** within the angle **BAE** has its symmetrical in **H'F'** (*905*). Thus, by taking three points, not in a straight line, in the angle **BAE**, we can show that their symmetricals are in the plane **B'A'E'**, and also in **A'E'D'**. In like manner, all the angles of **A'B'C'D'E'** can be shown to be in the same plane.

99. Cor.—*If two planes intersect, their symmetricals intersect, and the two intersections are symmetrical right lines.*

The student should show how this grows out of the proposition.

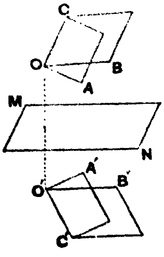


Fig. 466.

910. Theo.*—The symmetrical of a diedral is an equal diedral.*

Dem. **AOB** being the measure of the diedral **A-OC-B**, and **A'-O'C'-B'** the symmetrical diedral, and **O'** the symmetrical of **O**, the symmetrical of **AO** being **A'O'**, the angle **A'O'C'** is right, and in like manner **B'O'** being the symmetrical of **BO**, **B'O'C'** is right. But **BOA** = **B'O'A'** (?), whence the diedrals are equal.

911. Theo.—*Two polyedrons, symmetrical with respect to a plane, have their faces equal, each to each, and their homologous solid angles symmetrical.*

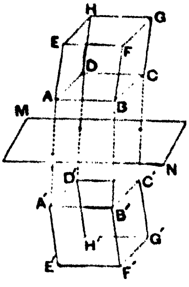


FIG. 467.

SUG'S.—This is an immediate consequence of preceding propositions. Thus E' being the symmetrical solid homologous with E, the homologous plane faces including them are equal (*908*). Again, the facial angles being equal, but not similarly disposed, the solid angles are symmetrical.

912. COR.—*Two symmetrical polyedrons can be decomposed into the same number of tetraedrons, symmetrical each to each.*

For we can decompose one of the polyedrons into tetraedrons having for their common vertex one of the vertices of this polyedron, and each of these tetraedrons will have its symmetrical in the other.

---

913. Theo.—*Two symmetrical polyedrons are equivalent.*

DEM.—From the last corollary it will appear that it is sufficient for the demonstration of this proposition to show that two symmetrical tetraedrons are equivalent (?). Let S-ABC, and S-ABC be two tetraedrons symmetrical with respect to their common base. They have a common base and equal altitudes (?), hence they are equivalent.

914. GENERAL SCHOLIUM.—We may speak of two loci, or two parts of the same locus, as symmetrical with respect to a line or plane, whenever all the points in one have symmetrical points in the other, even though the line joining the symmetrical points be not *perpendicular* to the axis, or the plane, of symmetry; observing, however, that this line is always bisected by the axis or plane. Thus, the ellipse in the figure is symmetrically divided by the line X'X, since every point in one portion has a symmetrical point in the other, as $Ps = P's$, for every point in the curve. In such a case the parts cannot be brought into coincidence by simple revolution: one part must be reversed.

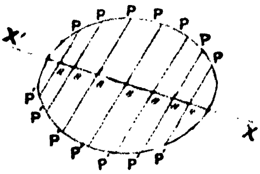


FIG. 468.

SECTION III.

OF MAXIMA AND MINIMA.

915. DEF.—*A* ***Maximum*** value of a magnitude conceived to vary continuously in some specified way, is a value which is greater than the preceding and succeeding values of the magnitude.

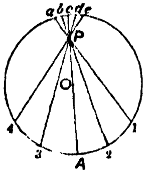


FIG. 469.

ILL'S.—Thus, suppose in a given circle, a chord passing through a fixed point, P, revolves so as to take successively the positions $1a$, $2b$, Ac, $3d$, $4e$, etc. It is at a maximum when it passes through the centre, as Ac. The chord is the magnitude which is conceived to vary in the way specified, and Ac is a value greater than the preceding and the succeeding values. Again, conceive a circle to be compressed or extended, as in the direction *mn*, so as to take the forms indicated by the dotted lines, its *area* will be diminished, the perimeter remaining the same. That is, of all figures of a given perimeter, the circle has the maximum area.

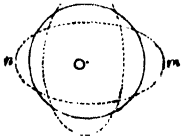


FIG. 470.

916. DEF.—*A* ***Minimum*** value of a magnitude conceived to vary continuously in some specified way, is a value which is less than the preceding and succeeding values of the magnitude.

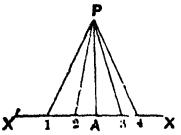


FIG. 471.

ILL'S.—Thus, conceive the varying magnitude to be a straight line from the fixed point P to the fixed line X'X; that is, suppose such a line to start from some position P1, and move through the successive positions P2, PA, P3, P4. PA is a minimum, since it is less than the preceding and succeeding values.

PROPOSITIONS CONCERNING MAXIMA AND MINIMA.

917. ***Axiom.***—*The minimum distance between two points is a straight line.*

---

918. ***Theo.***—*The minimum distance from a point to a line is a straight line perpendicular to the given line.*

Student give proof.

---

919. ***Theo.***—*The maximum line which can be inscribed in a given circle is a diameter.*

Proof based on the fact that the hypotenuse of a right angled triangle is the greatest side.

---

920. *Theo*—*The sum of the distances from two points on the same side of a line, to a point in the line, all being in the same plane, is a minimum when the lines measuring the distances make equal angles with the given line.*

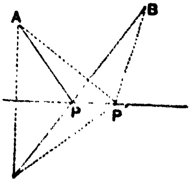


FIG. 472.

Student prove $AP + BP < AP' + BP'$.

---

921. *Theo.*—*If a triangle have a constant base and altitude, its vertical angle is a maximum when the triangle is isosceles.*

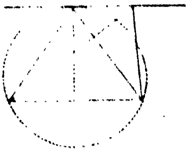


FIG. 473.

SUG.—By what is the vertical angle measured?

---

922. *Theo.*—*The base and area of a triangle being constant, its perimeter is a minimum when the triangle is isosceles.*

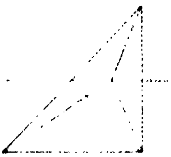


FIG. 474.

SUG'S.—The area and base being constant, the vertex remains in a line parallel to the base, for all values of the other sides. The figure will suggest the demonstration, which is based on the fact that any side of a triangle is less than the sum of the other two.

---

923. *Theo.*—*The difference between the distances from two points on opposite sides of a fixed line to a point in that line, is a maximum, when the lines measuring these distances make equal angles with the fixed line.*

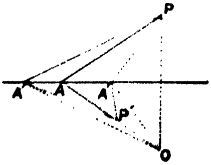


FIG. 475.

SUG'S. $P'O = AP - AP'$; but $P'O > A'O (= A'P) - A'P'$.

QUERY.—Having the points P, P', and the fixed line given, how is the point A found by geometrical construction?

---

924. *Theo.*—*The lengths of two sides of a triangle being constant, the area is a maximum when the included angle is right.*

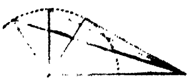


FIG. 476.

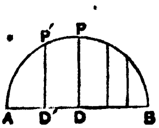


FIG. 477.

925. Theo.—*The sum of two adjacent sides of a rectangle being constant* (AB), *the area is a maximum when the sides are equal.*

ISOPERIMETRY.

926. *Isoperimetric* Figures are such as have equal perimeters, *i. e.*, bounding lines of equal length.

Problems in isoperimetry are a species of problems in Maxima and Minima. Thus, of all figures whose perimeters are *m* (say 10 inches), to find that which has the greatest area, is a problem in isoperimetry. Again, what must be the form of a pentagon whose perimeter is *m*, in order that its area may be a maximum?

---

927. Theo.—*Of isoperimetric triangles with a constant base, the isosceles is a maximum.*

Sug's.—By means of the figure to Theorem (***921***), we can readily show that any triangle having the same base as the isosceles triangle, and its vertex either in or beyond the line through the vertex of the isosceles triangle and parallel to its base, has a greater perimeter than the isosceles triangle. Hence, the isoperimetric triangle on the given base has its vertex below this parallel, except when isosceles; and consequently the isosceles is the maximum.

928. Cor.—*Of isoperimetric triangles, the equilateral has the maximum area* (?)

---

929. Prob.—*Given any triangle with a constant base, to construct the maximum isoperimetric triangle.*

930. Prob.—*Given any triangle, to construct the maximum isoperimetric triangle.*

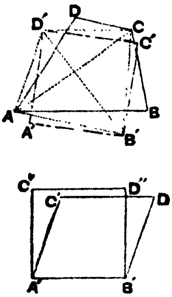


FIG. 478.

931. Theo.—*Of isoperimetric quadrilaterals, the square has the maximum area.*

Dem.—Let ABCD be any quadrilateral. If AD is not equal to DC, ADC can be replaced by the isosceles isoperimetric triangle AD'C, and the area of the quadrilateral increased. So ABC can be replaced by AB'C. Therefore AB'C'D > ABCD. In like manner if AD' is not equal to AB', D'AB' can be replaced by the maximum isoperimetric triangle D'A'B'. So also D'CB' can be replaced by D'C'B'. Therefore A'B'C'D' > AB'CD' > ABCD. Now, A'B'C'D' is a rhombus (?), and the student can show that the square on A'B' is greater than *any* rhombus with the same side.

932. Prob.—*Having given a quadrilateral, to construct the maximum isoperimetric quadrilateral.*

---

933. Theo.—*Of isoperimetric quadrilaterals with a constant base, the maximum has its three remaining sides equal each to each, and the angles which they include equal.*

DEM.—Let ABCD be the maximum isoperimetric quadrilateral on the base AD, then $AB = BC = CD$, and angle $ABC = BCD$. For, if AB is not equal to BC, draw AC, and replacing the triangle ABC with its isoperimetric isosceles triangle, we shall have a quadrilateral isoperimetric with ABCD, and greater than ABCD, *i. e.*, greater than the maximum, which is absurd.

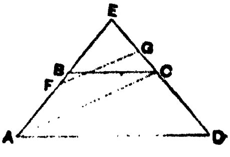


FIG. 479.

Again, if angle ABC is not equal to BCD, let $ABC < BCD$, whence $BCE < EBC$, and $BE < EC$. Take $EF = EC$, and $EG = EB$, whence the triangles FEG and BEC are equal, and $FG = BC$. Also, since $AB + BC + CD = AE + ED - (EB + EC) + BC$, and $AF + FG + GD = AE + ED - (FE + EG) + FG$, it follows that AFGD and ABCD are isoperimetrical, and, since $ABCD = AED - BEC$, and $AFGD = AED - FEG$, that AFGD and ABCD are equal. Therefore, AFGD is a maximum, and by the preceding part of the demonstration $AF = FG = BC = AB$, which is absurd; and there can be no inequality between angles ABC and BCD.

---

934. Theo.—*Of isoperimetric polygons of a given number of sides, the regular polygon has the maximum area.*

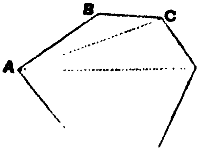


FIG. 480.

DEM.—First, the polygon must be equilateral; for, if any two adjacent sides, as AB, BC, are unequal, the triangle ABC can be replaced by its isoperimetric isosceles triangle, and thus the area of the polygon be increased.

Second, the polygon must be equiangular; for, if any two adjacent angles, as B and C, are unequal, the quadrilateral ABCD can be replaced by its isoperimetric quadrilateral with $B = C$, and thus the area of the polygon be increased.

935. Theo.—*Of isoperimetric regular polygons, the one of the greater number of sides is the greater.*

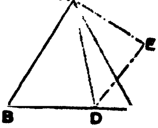


FIG. 481.

DEM.—Let ABC be an equilateral (regular) triangle. Join any vertex, as A, with any point, as D, in the opposite side. Replace the triangle ACD with the isosceles isoperimetric triangle AED. Then is the quadrilateral $ABDE >$ the triangle ABC.

But, of isoperimetric quadrilaterals, the regular (the square) is the greater. Hence, the regular quadrilateral (the square) isoperimetric with the triangle ABC, is greater than the triangle. In the same manner the regular pentagon isoperimetric with the square can be shown greater than the square; and thus on, *ad libitum*.

936. COR.—*Of plane isoperimetric figures, the circle has the maximum area,* since it is the limiting form of the regular polygon, as the number of its sides is indefinitely increased.

SECTION IV.

OF TRANSVERSALS.

937. DEF.—***A Transversal*** is a line cutting a system of lines. A transversal of a triangle is a line cutting its sides;* it either cuts two sides and the third side produced, or the three sides produced. In speaking of the transversal of a triangle (or polygon), the distances on any side (or side produced) from the intersection of the transversal with that side to the angles, are *Segments.* Of these there are six. *Adjacent segments* are such as have an extremity of each at the same point. *Non-adjacent segments* are such as have no extremity common.

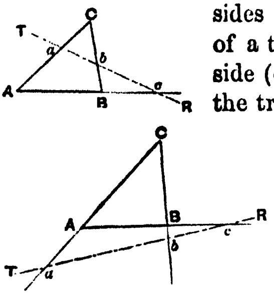


FIG. 482.

ILL'S. TR is a transversal of the triangle ABC; *a*A, *a*C, *b*C, *b*B, *c*A, *c*B are adjacent segments two and two; *a*C, *b*B, *c*A, and *a*A, *b*C, *c*B are the two groups of non-adjacent segments.

938. THE TWO FUNDAMENTAL PROPOSITIONS OF THE THEORY OF TRANSVERSALS.

939. *Theo.*—*The product of three non-adjacent segments of the sides of a triangle cut by a transversal, is equal to the product of the other three.*

DEM.—ABC being cut by the transversal TR, $a\text{A} \times b\text{C} \times c\text{B} = a\text{C} \times b\text{B} \times c\text{A}$. Draw BD parallel to AC, and from the similar triangles we have

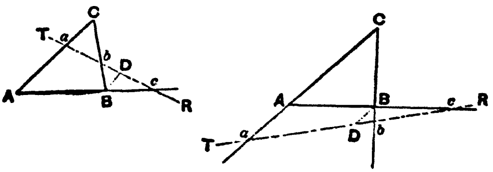


FIG. 483.

$$\frac{a\text{A}}{\text{BD}} = \frac{c\text{A}}{c\text{B}}, \text{ and } \frac{\text{DB}}{b\text{B}} = \frac{a\text{C}}{b\text{C}};$$

whence, multiplying,

$$\frac{a\text{A}}{b\text{B}} = \frac{a\text{C} \times c\text{A}}{b\text{C} \times c\text{B}},$$

or $a\text{A} \times b\text{C} \times c\text{B} = a\text{C} \times b\text{B} \times c\text{A}$.

---

* Or, sides produced—this expression being usually omitted in higher Geometry as all lines are to be considered indefinite unless limited in the problem.

940. Cor.—Conversely, *If three points be taken in the sides of a triangle (as* a, b, c*) such that the product of three non-adjacent segments equals the product of the other three, the points are in the same straight line.*

For, passing a line through a and b, let it cut the third side in c'. Then, by the proposition, $aA \times bC \times c'B = aC \times bB \times c'A$. But, by hypothesis $aA \times bC \times cB = aC \times bB \times cA$. Whence $\frac{cB}{c'B} = \frac{cA}{c'A}$, and c and c' must coincide.

Sch.—This theorem is known among mathematicians as *The Ptolemaic Theorem*, and is usually attributed to Claudius Ptolemy, an Egyptian mathematician and philosopher who flourished in Alexandria during the first half of the second century. But it is thought to be more properly due to *Menelaus*, who lived a century before Ptolemy.

941. ***Theo.***—*The three angle-transversals* of a triangle, passing through a common point, divide the sides into segments such that the product of three non-adjacent segments equals the product of the other three.*

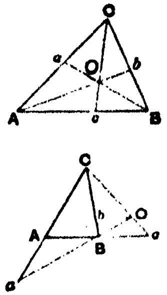


Fig. 484.

Dem.—From the triangle ACc cut by the transversal aB, we have $aA \times CO \times cB = AB \times aC \times Oc$; and from CB$c$ cut by bA, $Oc \times bC \times AB = CO \times bB \times cA$. Multiplying, we obtain $aA \times bC \times cB = aC \times bB \times cA$.

942. Cor. 1.—Conversely, *If the three angle-transversals of a triangle divide the sides into segments such that the product of three non-adjacent segments equals the product of the other three, the transversals pass through a common point.*

For, the sides being divided at a, b, and c, so that $aA \times bC \times cB = aC \times bB \times cA$, draw C$c$, and A$b$, and let O be their intersection. Now, let a' be the point in which BO cuts AC. Then, by the proposition, $a'A \times bC \times cB = a'C \times bB \times cA$.

Whence $\frac{aA}{a'A} = \frac{aC}{a'C}$, and a and a' coincide.

943. Cor. 2.—*If any one of the sides is bisected, the line joining the other points of division is parallel to this side.*

For, let $bC = bB$. Then $aA \times bC \times cB = aC \times bB \times cA$, becomes
$aA \times cB = aC \times cA$; or $Aa : aC :: Ac : cB$.

Query.—How does this apply to the second figure?

944. Cor. 3.—*If the line joining two points of division is parallel to the third side, the latter side is bisected.*

---

* The transversals passing through the angles.

For, if ab is parallel to AB, $aC : bC :: aA : bB$, whence $aC \times bB = bC \times aA$. And, since $aA \times bC \times cB = aC \times bB \times cA$, $cA = cB$.

---

945. We will now give a few problems to illustrate the use of the theory of transversals.

946. Prob.*—To show that the medial lines of a triangle pass through a common point.*

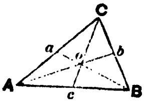


FIG. 485.

SOLUTION.—Since $aA = aC$, $bC = bB$, and $cB = cA$, by multiplying, we have $aA \times bC \times cB = aC \times bB \times cA$; whence by the last corollary these transversals pass through a common point.

---

947. Prob.*—To show that the bisectors of the angles of a triangle pass through a common point.*

SOLUTION.—In the last figure let aB, bA, cC be the bisectors.

Then $\frac{aA}{aC} = \frac{AB}{CB}$, $\frac{bC}{bB} = \frac{AC}{AB}$, $\frac{cB}{cA} = \frac{CB}{AC}$; multiplying, $\frac{aA \times bC \times cB}{aC \times bB \times cA} = 1$, or $aA \times bC \times cB = aC \times bB \times cA$. Therefore these transversals pass through a common point.

---

948. Prob.*—To show that the altitudes of a triangle pass through a common point.*

SUG'S.—In the last figure, if aB, bA, cC, were the perpendiculars, there would be three pairs of similar triangles giving $\frac{aA}{bB} = \frac{AO}{BO}$, $\frac{bC}{cA} = \frac{CO}{AO}$, $\frac{cB}{aC} = \frac{OB}{CO}$; whence, as in the last.

---

949. Prob.*—To show that the angle-transversals terminating in the points of tangency of the sides of the triangle with its inscribed circle, pass through a common point.*

SUG'S.—In the last figure, if a, b, c were the points of tangency we should have $aA = cA$, $bC = aC$, $cB = bB$; whence $aA \times bC \times cB = aC \times bB \times cA$. Which shows that the transversals pass through a common point.

950. Theo.*—If two sides of a triangle are divided proportionally, starting from the vertex, the angle-transversals from the extremities of the other side to the corresponding points of division, intersect in the medial line to this third side.*

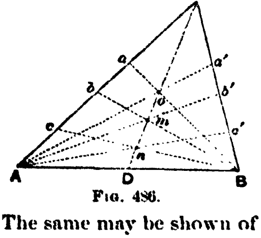


FIG. 486.

DEM.—Since AC and CB are divided proportionally at a and a', $aA \times a'C = aC \times a'B$; and as $DB = DA$, $aA \times a'C \times DB = aC \times a'B \times DA$, the angle-transversals Aa', Ba intersect in CD. The same may be shown of any other angle-transversals from A and B, dividing CB and CA proportionally.

951. Cor.—*In any trapezoid the transversal passing through the intersection of the diagonals, and the intersection of the non-parallel sides, bisects the parallel sides.*

Sug.—Joining aa' in the last figure, CD is such a transversal. The student will readily see the connection with the proposition.

952. Prob.—*Through a given point to draw a line which shall meet two given lines at their intersection in an invisible, inaccessible point.*

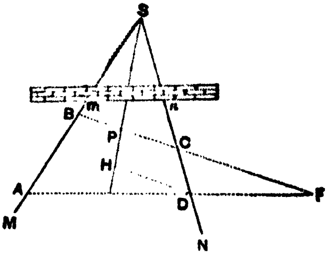


Fig. 487.

Solution.—Let Mm, Nn be the two given lines which meet in the invisible, inaccessible point S, and P the given point through which a line is to be located which will meet Mm, Nn in S. Through P draw any convenient transversal, as BF, and any other meeting this, as AF. Now, considering MS as a transversal of the triangle CDF, we have $AF \times BC \times SD = AD \times BF \times SC$; whence $\frac{SD}{SC} = \frac{AD \times BF}{AF \times BC}$. But, HD being drawn parallel to BF, we have $\frac{HD}{PC} = \frac{SD}{SC} = \frac{AD \times BF}{AF \times BC}$, or $HD = \frac{AD \times BF \times PC}{AF \times BC}$; whence HD is known, as AD, BF, PC, AF, BC can be measured. The points P and H determine the required line.

953. Def.—The ***Complete Quadrilateral*** is the figure formed by four lines meeting in six points. The complete quadrilateral has three diagonals.

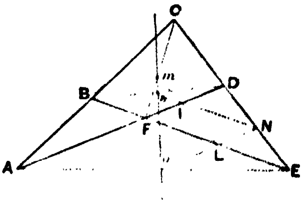


Fig. 488.

Ill.—ABCDEF is a complete quadrilateral, and its diagonals are CF, BD, and AE, the latter being spoken of as the third or exterior diagonal.

954. Theo.—*The middle points of the three diagonals of a complete quadrilateral are in the same straight line.*

Dem.—m, n, o being the centres of the diagonals of the complete quadrilateral, in the preceding figure, are in the same straight line. Bisect the sides of the triangle FDE, as at I, N, L, and draw IN, IL, LN. Since IN is parallel to BE, and bisects DF and DE, it also bisects DB (?) and hence passes through n. For like reasons IL passes through m, and LN through o. Now, AC being a transversal of the triangle FDE gives $CD \times BE \times AF = CE \times BF \times AD$. Therefore, noticing that $\frac{1}{2}\text{CD} = m\text{I}$, $\frac{1}{2}\text{BE} = n\text{N}$, $\frac{1}{2}\text{AF} = o\text{L}$, $\frac{1}{2}\text{CE} = m\text{L}$, $\frac{1}{2}\text{BF} = n\text{I}$, and $\frac{1}{2}\text{AD} = o\text{N}$, we have $m\text{I} \times n\text{N} \times o\text{L} = m\text{L} \times n\text{I} \times o\text{N}$. Hence these three points m, n, o lie in a transversal to the triangle ILN.

SECTION V.

HARMONIC PROPORTION AND HARMONIC PENCILS.

955. Def.—Three quantities are in *Harmonic Proportion* when the difference between the first and second is to the difference between the second and third, as the first is to the third.

Ill.—6, 4, 3 are in harmonic proportion, since $6 - 4 : 4 - 3 :: 6 : 3$. In general, a, b, c are in harmonic proportion, if $a - b : b - c :: a : c$.

956. ***Theo.***—*If a given line be divided internally and externally in the same geometric ratio, the distance between the points of division is a harmonic mean between the distances of the extremities of the given line from the point not included between them.*

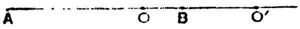


Fig. 490.

Dem. — Let AB be the given line; and let O and O′ be so taken that $\text{AO} : \text{BO} :: \text{AO}' : \text{BO}'$; then is OO′ a harmonic mean between AO′ and BO′. For $\text{AO} = \text{AO}' - \text{OO}'$, and $\text{BO} = \text{OO}' - \text{BO}'$; whence AO′, OO′, and BO′ are such that $\text{AO} - \text{OO}' : \text{OO}' - \text{BO}' :: \text{AO}' : \text{BO}'$, that is, they are in harmonic proportion.

957. Cor. 1.—*AO, AB, and AO′ are in harmonic proportion,* i.e., *AB is a harmonic mean between AO and AO′.*

For $\text{AB} - \text{AO}\ (= \text{BO}) : \text{AO}' - \text{AB}\ (= \text{BO}') :: \text{AO} : \text{AO}'$.

958. Cor. 2.—*When AO, OO′, BO′ are in harmonic proportion, $\text{AO} \times \text{BO}' = \text{BO} \times \text{AO}'$.*

959. Cor. 3.—Conversely, *When a line is divided into three parts such that the rectangle of the extreme parts equals the rectangle of the mean part into the whole line, the line is divided harmonically.*

Thus, let AO′ be the line, and $\text{AO}' \times \text{BO}' = \text{BO} \times \text{AO}'$; then $\text{AO} : \text{BO} :: \text{AO}' : \text{BO}'$, whence, by the proposition, OO′ is a harmonic mean between AO′ and BO′.

Def.—The points O and O′ are called *Harmonic Conjugates.*

960. Theo.—*If two lines be drawn, one bisecting the interior and the other the adjacent exterior angle of a triangle, and meeting the opposite side,* they divide this line harmonically.*

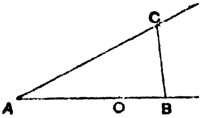


FIG. 491.

SUG.—By means of (*358, 359*, PART II.) the student will be enabled to establish the relation $AO : BO :: AO' : BO'$, whence, by the last proposition, AO', OO', BO' are in harmonic proportion.

961. Theo.—*In the complete quadrilateral, any diagonal is divided harmonically by the other two.*

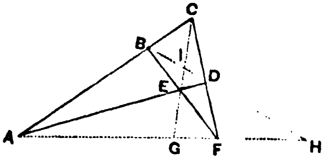


FIG. 489.

DEM.—Thus, AFH is divided harmonically at G and H. For, considering BH as a transversal of the triangle ACF, we have $HF \times DC \times BA = HA \times DF \times BC$. And CG, AD, FB being angle-transversals of the same triangle, we have $GF \times BA \times DC = GA \times BC \times DF$. Whence, dividing, $\frac{HF}{GF} = \frac{HA}{GA}$, *i.e.*, AH is divided harmonically. Again, if CH is drawn, CA, CG, CF, CH constitute a harmonic pencil, and BH, a transversal of it, is cut harmonically at B, I, D, H. Finally, if F and I be joined, FH (or FA), FB, FI, FD constitute a harmonic pencil, and hence CG is cut harmonically at C, I, E, G.

962. COR.—An angle-transversal of a triangle, and a line passing through the feet of the other angle-transversals, divide the third side harmonically.

963. Prob.—*Given a right line to locate two harmonic conjugate points.*

SOLUTION.—Let AB be the line. O may be taken at pleasure between A and B. We are then to find O′, so that $AO : BO :: AO' : BO'$. Taking this by division, we have $AO - BO : BO :: AO' - BO' (= AB) : BO'$. The first three terms being known, the other can be constructed. Or, we may first locate O′ at pleasure, and then find O.

964. Theo.—*If from the given point* C *in a line the distances* CO, CB, CO′ *be taken in the same direction, so that* $CO \times CO' = CB^2$; *and if* $CA = CB$ *be taken in the opposite direction,* AO′ *will be divided harmonically at* O *and* B.

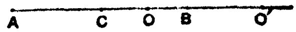


FIG. 492.

DEM. — From $CO \times CO' = CB^2$, we readily write $CO : CB :: CB : CO'$, $CB + CO (= AO) : CB - CO (= BO) :: CO' + CB (= AO') : CO' - CB (= BO')$.

---

* The bisector of the exterior angle meets the side *produced* ; but in higher geometry, as it is always understood that lines are indefinite unless limited by hypothesis, such specifications are deemed unnecessary.

965. Cor. 1.—Conversely, *If a line* AO' *be cut harmonically at* O *and* B, *and either of the harmonic means be bisected, as* AB *at* C, *the three segments* CO, CB, CO' *will be in geometric proportion.*

For, since $AO' : BO' :: AO : BO$, $AO' + BO' : AO' - BO' :: AO + BO : AO - BO$, or $2CO' : 2CB :: 2CB : 2CO$, and $CO' : CB :: CB : CO$.

966. Cor. 2.—In a given line, as AB, as O approaches the centre C, O' recedes, and when O is at C, O' is at infinity, since $CO' = \frac{\overline{CB}^2}{CO}$.

967. ***Theo.***—*The geometric mean between two lines is also the geometric mean between their arithmetic and harmonic means.*

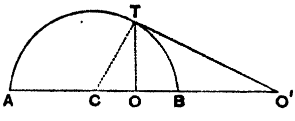


Fig. 493.

Dem.—Let AO' and BO' be the two lines. On their difference, AB, draw a semicircle, draw the tangent $O'T$ and let fall the perpendicular TO. Then O and O' are harmonic conjugates, since $CO \times CO' = \overline{CB}^2$ (?), CO' is the arithmetic mean (that is, $\frac{1}{2}$ the sum) of AO' and BO' (?) and TO' is the geometric mean (?). $\therefore CO' : TO' :: TO' : OO'$ (?).

Queries.—Which is the greatest, in general, the arithmetic, geometric, or harmonic mean between two quantities? Are they ever equal?

968. Sch.—*This proposition* affords a ready method of finding either of the harmonic conjugates O or O', when the other is given. The student will show how.

969. Cor. 1.—*The rectangle of the harmonic means and the sum of the extremes, is equivalent to twice the rectangle of the extremes.*

For, $CO' \times OO' = \overline{TO'}^2 = AO' \times BO'$, whence $2CO' \times OO' = 2AO' \times BO'$; and, since $2CO' = AO' + BO'$, $(AO' + BO') \times OO' = 2AO' \times BO'$.

970. Cor. 2.—*The rectangle of the harmonic mean and the difference of the differences of the* 1st *and* 2nd, *and the* 2nd *and* 3rd, *is equivalent to twice the rectangle of these differences.*

That is, $OO' \times [(AO' - OO') - (OO' - BO')] = 2\,(AO' - OO')\,(OO' - BO')$, or $OO' \times (AO - BO) = 2AO \times BO$. Let the student give the proof.

971. Cor. 3.—*If three quantities are in harmonic proportion their reciprocals are in arithmetic proportion* (*i.e.*, the difference between the 1st and 2nd equals the difference between the 2nd and 3rd).

For, from AO', OO', BO', we have the reciprocals $\frac{1}{AO'}$, $\frac{1}{OO'}$, $\frac{1}{BO'}$. Now $\frac{1}{OO'} - \frac{1}{AO'} = \frac{AO' - OO'}{OO' \times AO'} = \frac{AO}{OO' \times AO'}$; and $\frac{1}{BO'} - \frac{1}{OO'} = \frac{OO' - BO'}{OO' \times BO'}$

$= \frac{\text{BO}}{\text{OO}' \times \text{BO}'}$. But $\frac{\text{AO}}{\text{OO}' \times \text{AO}'} = \frac{\text{BO}}{\text{OO}' \times \text{BO}'}$ since $\frac{\text{AO}}{\text{AO}'} = \frac{\text{BO}}{\text{BO}'}$ (?). $\therefore \frac{1}{\text{OO}'} - \frac{1}{\text{AO}'} = \frac{1}{\text{BO}'} - \frac{1}{\text{OO}'}$.

---

972. Prob.—*Given the harmonic mean and the difference between the extremes, to find the extremes.*

SUG'S.—We have OO′ and AB, (*Fig.* 493, Art. ***967***) given. Then $\text{CO} \times \text{CO}' = \overline{\text{CT}}^2 = \frac{1}{4}\overline{\text{AB}}^2$, and $\text{CO}' - \text{CO} = \text{OO}'$, whence $\overline{\text{CO}'}^2 - \text{OO}' \times \text{CO}' = \frac{1}{4}\overline{\text{AB}}^2$. From this equation CO′ can be constructed (***832***), and the problem solved.

---

973. Theo.—*When two circles cut each other orthogonally* (*i. e.*, so that the tangents at the common point are at right angles), *any line passing through the centre of one, and cutting the other, is divided harmonically by the circumferences.*

DEM.—The tangents being perpendicular to each other pass through the centres, hence $\text{CO} \times \text{CO}' = \overline{\text{CT}}^2$. But CB = CT. Therefore AO′ is cut harmonically.

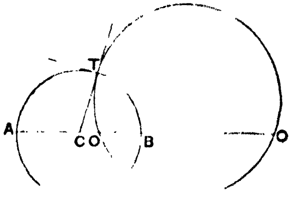


FIG. 494.

---

974. Prob.—*To find the altitude of a triangle in terms of the radii of the escribed circles touching the adjacent sides.*

SOLUTION.—Let r and r' be the radii of the escribed circles, and p the altitude. Now RT, AT, and QT are in harmonic proportion; since, considering the triangle ACT, CQ bisects its interior and RC its exterior angle (?), we have QT : QA : : RT : RA. But r, p, r', sustain the same relation to each other as RT, AT, QT; hence r, p, r' are in harmonical proportion. Therefore, by (***469***) $p(r+r') = 2rr'$; or $p = \frac{2rr'}{r+r'}$.

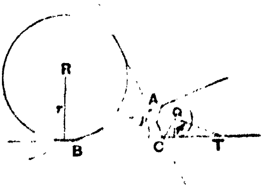


FIG. 495.

HARMONIC PENCILS.

975. DEF.—***A Pencil*** of lines is a series of lines diverging from a common point.

DEF.—***A Harmonic Pencil*** is a pencil of four lines cutting another line harmonically.

ILL.—In the following figure OA, OB, OC, OD constitute a *Harmonic Pencil*, if they divide the line *mn* harmonically at A, B, C, D.

***976. Theo.**—A harmonic pencil divides harmonically every line which cuts it.*

Dem.—OA, OB, OC, OD being a harmonic pencil, that is, AD, BD, CD, being in harmonic proportion, A'D', any other line cutting the pencil, is divided harmonically, so that A'D', B'D', C'D', are in harmonic proportion. Through C and C' draw parallels to OA, as LK and L'K'. Now, from similar triangles, AB : BC : : AO : CK, and AO : CL : : AD : CD. But AD : CD : : AB : BC, since AD is harmonically divided. Hence AO : CK : : AO : CL, and CK = CL. Hence from similar triangles C'K' = C'L'. Again A'B' : B'C' : : A'O : C'K' (?), and A'D' : C'D' : : A'O : C'L' (= C'K') (?), whence A'B' : B'C' : : A'D' : C'D', or A'D', B'D', C'D' are in harmonic proportion.

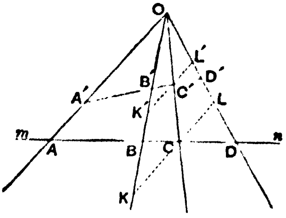


Fig. 496.

Sch.—If the line through C' cut the prolongation of AO beyond O, it is still harmonically divided; and, in fact, it is scarcely necessary to make this statement, since in all general discussions lines are to be considered indefinite, unless limited by hypothesis.

977. Def.—The alternate legs of a harmonic pencil are called *conjugate*, as OA and OC, OB and OD.

***978. Theo.**—If two conjugate legs of a harmonic pencil be at right angles, one of them bisects the angle included by the other pair, and the other the supplement of this angle.*

Sug.—This is the converse of (**962**), remembering that the bisectors of two adjacent supplemental angles are at right angles.

SECTION VI.

ANHARMONIC RATIO.

979. Def.—***The Anharmonic Ratio*** of four points in a right line is the ratio of the rectangle of the distance between the first and fourth into the distance between the second and third to the rectangle of the distance between the first and second into the distance between the second and fourth.

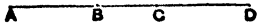


Fig. 497.

Ill.—The anharmonic ratio of the four points A, B, C, D is AD × BC : AB × CD.

980. The relation AD × BC : AB × CD is expressed for brevity [ABCD].

Ill.—Thus [ABCD] means AD × BC : AB × CD; [ADBC] means AC × DB : AD × BC; [BACD] means BD × AC : BA × CD; etc. The ratio [ABCD], or AD × BC : AB × CD is evidently the same as $\frac{AB}{BC} : \frac{AD}{CD}$.

981. Sch.—The appropriateness of the term *anharmonic* (*not*-harmonic) will be seen when we observe that, if AD is *harmonically* divided, $\frac{AB}{BC}$ *equals* $\frac{AD}{CD}$. If, therefore, $\frac{AB}{BC}$ is *not* equal to $\frac{AD}{CD}$, which is the *general* case of division, irrespective of the position of the points B and C, we may consider the *ratio* of $\frac{AB}{BC}$ to $\frac{AD}{CD}$. This general ratio, or, what is the same thing, AD × BC : AB × CD, is called the anharmonic ratio.

---

982. **Theo.**—*The anharmonic ratio of four points is not changed by interchanging two of the letters, provided the other two be interchanged at the same time.*

Dem. [ABCD] = [DCBA] = [BADC] = [CDAB], *i. e.*, AD × BC : AB × CD = DA × CB : DC × BA − BC × AD : BA × DC = CB × DA : CD × AB, which are evidently identical. [The student should notice the different segments of the line indicated by the different forms.]

983. Sch.—But [ACBD] is a *different* anharmonic ratio from [ABCD]; since AD × CB : AC × BD is not necessarily equal to AD × CB : AB × CD. Now, as there can be twenty-four permutations of four letters, there may be formed six different anharmonic ratios from four given points in a line.

---

984. **Theo.**—*If a pencil of four lines is cut by any transversal, the anharmonic ratio of the four points of intersection is constant.*

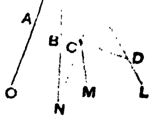


Fig. 498.

Dem. SL, SM, SN, SO, or, as we may read it, S-L,M,N,O, being such a pencil, and AD any transversal, draw through C NP parallel to SO. Then,

$$AD \times BC : AB \times CD :: \left\{ \begin{matrix} BC : AB \\ AD : CD \end{matrix} \right\} :: \left\{ \begin{matrix} CN : AS \\ AS : CP \end{matrix} \right\} :: CN : CP.$$

But CN : CP is constant for all positions of C on SM. Therefore AD × BC : AB × CD is constant for any transversal.

985. Sch.—Other constant ratios may be written from the preceding proposition and scholium. The anharmonic ratio [ABCD] is called the anharmonic ratio of the pencil. The angles of the pencil are the six angles included by the rays.

***986.*—Cor. 1.**—*If two pencils are mutually equiangular their anharmonic ratios are equal.*

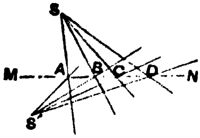


FIG. 499.

Query.—Is the converse of this corollary true?

987. Cor. 2.—*If two pencils have their intersections in the same right line, their anharmonic ratios are equal.*

988. Def.—The anharmonic ratio of four points on the circumference of a circle is the anharmonic ratio of the pencil formed by joining these points with any point in the circumference.

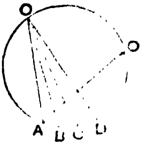


FIG. 500.

Ill.—Thus, the anharmonic ratio of the points **A**, **B**, **C**, **D** is the anharmonic ratio of the pencil **O-A,B,C,D**, it being immaterial where in the circumference the point **O** is taken, since by Cor. 1, preceding, the ratio is the same for any position of **O** (?).

989.* *Theo.—*If four fixed tangents to a circle are cut by a fifth, the anharmonic ratio of the four points of intersection, called the anharmonic ratio of the tangents, is constant.*

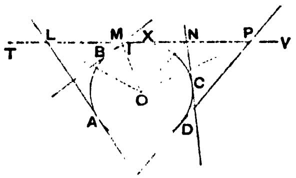


FIG. 501.

Dem. **A, B, C, D** being the fixed points of tangency, any transversal, as **TV**, cutting the tangents, has the anharmonic ratio [**LMNP**] constant. For the pencil **O-L,M,N,P** has its angles constant. Thus **LOM** is measured by ½ arc (**AX**—**BX**) = ½**AB**, which is constant. And in like manner **MON** is measured by ½ arc **BC**, and **NOP** is measured by ½ arc **CD**. Hence, by the first of the preceding corollaries, the anharmonic ratio [**LMNP**] is constant.

990. The theory of anharmonic ratio is applied with great facility to the demonstration of theorems showing that several points are in a right line, and that several lines intersect in a common point. We give three specimens of each class.

991.* *Theo.—*If two pencils have the same anharmonic ratio and a homologous ray common, the intersection of the other homologous rays are in the same right line.*

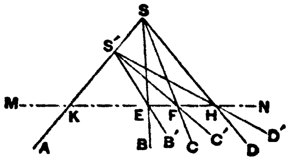


FIG. 502.

Dem.—Let **S-A,B,C,D** and **S'-A,B',C',D'** be two pencils having the same anharmonic ratio, and the rays SA, S′A coincident; then the intersections E, F, H are in the same right line. Let the line passing through E and F intersect SA in K, and suppose it intersect SD in H′, and S′D′ in H″. Then, since the anharmonic ratios of the two pencils are equal [KEFH′] = [KEFH″]; whence H′ and H″ are the same point, and must be the intersection of the two lines SD, S′D′, that is, H.

---

992. Theo.—*If in two right lines four points in the one have the same anharmonic ratio as four points in the other, and one homologous point in common, the three lines passing through the other pairs of homologous points meet in a common point.*

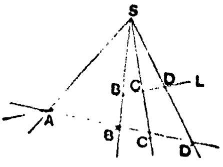


Fig. 503.

Dem.—Let A be common, and [ABCD] = [A′B′C′D′]. Draw SA and SD′. Call the point in which SD′ cuts AL D″ (for the time being). Then [AB′C′D′] = [ABCD″]. But by hypothesis [AB′C′D′] = [ABCD]. Therefore [ABCD] = [ABCD″], and D and D″ are one and the same point. Hence the three lines which pass through B and B′, C and C′, D and D′ meet in a common point S.

---

993. Theo.—*If the lines passing through the corresponding vertices of two triangles meet in a common point, the intersections of their homologous sides lie in the same right line.*

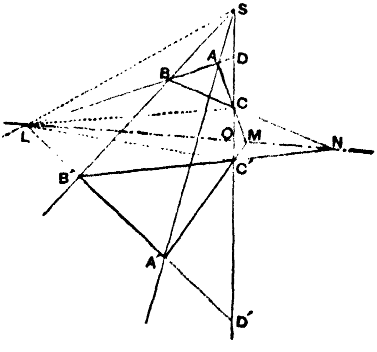


Fig. 504.

Dem.—Let ABC and A′B′C′ be two triangles so situated that the lines AA′, BB′, CC′ meet in the common point S; then L, M, N, the intersections of the homologous sides, are in a right line. For the pencil S-L, B, A, C being cut by the two transversals LD, LD′, gives [LBAD] = [LB′A′D′] (*984*). But C-L,B,A,D, and C′-L,B′,A′,D′, have these anharmonic ratios, hence C-L, O,M,N, and C′-L,O,M,N, their equivalents, and having a common ray CC′, have equal anharmonic ratios, and consequently L, M, N are in the same right line (*991*).

---

994. Theo.—*If the intersection of the corresponding sides of two triangles are in the same right line, the lines passing through their corresponding angles meet in a common point.*

Dem.—In the last figure, if AB and A′B′, AC and A′C′, BC and B′C′ have their intersections in the same right line, as **LN**, the lines passing through **B** and **B′**, **A** and **A′**, **C** and **C′** meet in a common point, as **S**. By (*987*) **C-L,O,M,N** has the same anharmonic ratio as **C′-L,O,M,N**, whence [**LBAD**] = [**LB′A′D′**], and the truth of the theorem follows from (992).

995. Theo.*—The opposite sides of an inscribed hexagon have their intersections in the same straight line.*

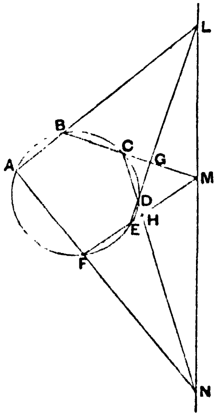


Fig. 505.

Dem.—The anharmonic ratios of the pencils **B-A,E,D,C,*** and **F-A,E,D,C** being equal (*988*), **LCDE**, which intersects the first, is divided in the same anharmonic ratio as **NHDC**, which cuts the second, or [**LCDE**] = [**NCDH**]. But these lines have a common homologous point **D**, hence the lines joining the other pairs of homologous points, as **LN**, **CC**, **EH**, meet in a common point, as **M**. Therefore **L**, **M**, **N** are in the same right line.

996. Sch.—This theorem is due to Pascal, whose wonderful achievements in his brief life of thirty-nine years (1623–1662) have been the admiration of all succeeding generations.

Theo.*—The diagonals joining the opposite vertices of a circumscribed hexagon intersect in a common point.*

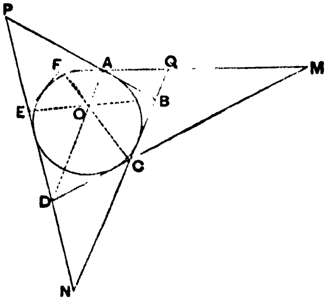


Fig. 506.

Dem.—Consider **AB**, **BC**, **CD**, **EF** four fixed tangents cut by **ED** and **FA**. Then [**PNDE**] = [**AQMF**] (*989*). Hence the anharmonic ratios of **B-P,N,D,E,*** and **C-A,Q,M,F** are equal (*985*); and since they have a common ray (**CQ**, **BN**) the intersections **A**, **O**, **D**, of their homologous rays, are in the same right line. Therefore the diagonals pass through a common

---

* The student can *conceive* the rays **BE**, **BD**, etc., as drawn, without encumbering the figure ith them.

SECTION VII.

POLE AND POLAR IN RESPECT TO A CIRCLE.

998. DEF.—If a secant to a circle be revolved about a fixed point in the plane of the circle, the locus of the harmonic conjugate of the fixed point, in reference to the intersections, is the ***Polar*** of the fixed point. The fixed point is the ***Pole*** of the *Polar Line*. The terms pole and polar as here used are correlative, and neither has any significance without the other.

ILL.—Let AP be a secant revolving about the fixed point P, and let C be so taken that (in every position) AC : CB : : AP : BP, then is the locus of C the *Polar* of P, and P is the *Pole* of the locus of C.

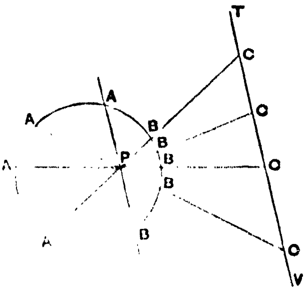


FIG. 507.

999. ***Theo.***—*The Polar of a given point in respect to a circle is a right line.*

DEM.—Let P be the pole, AP any secant passing through P, and a point in the polar. The locus of C is required. Draw PL through the centre, and let fall the perpendicular CC'. Draw AL, AH, AC'F, and C'B. Since AC : CB : : AP : BP, C'P bisects the angle BC'F, the exterior angle of the triangle AC'B (?); hence, as LAH is a right angle, AL bisects NAC', the exterior angle of the triangle C'AP (?). Therefore, PL is harmonically divided at C', and H; and, C' being a fixed point, and C any point in the locus, the locus is the perpendicular TCC'V.

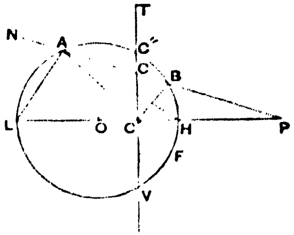


FIG. 508.

1000. COR. 1.—*Since, as the secant revolves, the points* A *and* B *will vanish in* C'', C'' *is the point at which a tangent from the pole* P *touches the circle.*

1001. COR. 2.—*Drawing* OC'', C''P, *we see that* $\overline{OC''}^2$ *(or* $\overline{OH}^2$*)* = OC' × OP.

1002. COR. 3.—*The polar of a point in the circumference is a tangent at that point.*

For, as OC′ × OP is constant and equal to $\overline{OH}^2$, OP diminishes as OC′ increases, and when OP = OH, OC′ = OH also.

1003. Prob.—*To draw the polar to a given pole in respect to a given circle.*

COR. 1 effects the solution.

1004. Prob.—*To find the pole of a polar to a given circle.*

Through the centre draw a perpendicular to the polar. [The student should be able to complete the solution.]

1005. DEF.—The point C′ where the polar cuts the line passing through the pole and the centre of the circle is called the *Polar Point.*

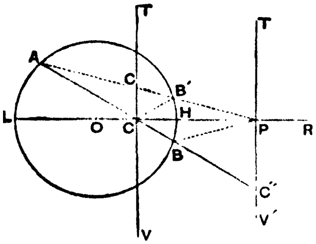


FIG. 509.

1006. Theo.—*The pole and polar point are interchangeable.*

DEM.—TV being the polar to P, we are to show that T′V′, parallel to TV and passing through P, is polar to C′; *i.e.*, that any secant, as AC′C″, passing through C′, is divided harmonically in the intersections with the circumference, C′, and the intersection with T′V′. Drawing AP, since P is the pole of TV, we have, as in the last demonstration, angle APB bisected by PC′; and consequently RPB bisected by PC″. Therefore AC′ : C′B :: AC″ : BC″. Q. E. D.

1007. Theo.—*The polars of all the points in a right line pass through the pole of that line; and, conversely, The poles of all straight lines which pass through a given point are in the polar of that point.*

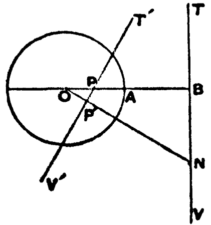


FIG. 510.

DEM.—1st. TV being a given line and P its pole, we are to show that the polar to any point, as N, passes through P. Draw through P a perpendicular to ON; then P′ is the polar point to N. For, OP : ON :: OP′ : OB (?): whence ON × OP′ = OP × OB = $\overline{OA}^2$. Therefore, T′V′ is the polar of N (?). 2nd. P being any point and TV its polar, the pole of any line, as T′V′ passing through P, is in TV, as at N. Draw ON perpendicular to T′V′. Then, as before, $ON \times OP' = OP \times OB = \overline{OA}^2$, and N is the pole of T′V′.

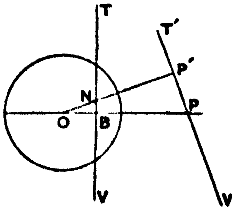


FIG. 511.

1008. Cor.—*The pole of a straight line is the intersection of the polars of any two of its points;* and, conversely, *The polar of any point is the straight line joining the poles of any two straight lines passing through that point.*

RECIPROCAL POLARS.

1009. Def.—If two polygons be constructed, one within, or inscribed in, a circle, and the other without, or circumscribed about the same circle, such that the vertices of the one are the poles of the sides of the other, the two polygons are called ***Reciprocal Polars***; and the circle is called the *Auxiliary Circle.*

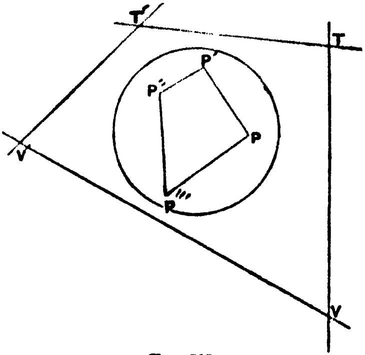


FIG. 512.

The possibility of constructing such polygons is apparent from the last theorem. When the points P, P′, P″, P‴ are in the circumference, TV, TT′, T′V′, V′V become tangents, as appears from (1002).

1010. ***Prob.***—*Having given one of two reciprocal polars, to construct the other.*

The student should be able to make the construction.

1011. By means of the relation between reciprocal polars a large class of propositions relating to the relative positions of lines and points, become, as it were, double; *i.e.*, one proposition being proved, another can be inferred. The process by which the inference is made is called *reciprocation.* We will give an example.

***1012. Prob.**—To deduce the reciprocal of Pascal's theorem* (995).

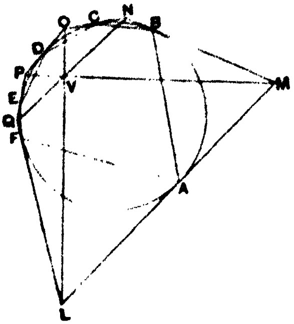


FIG. 513.

SOLUTION.—Draw tangents at the six vertices of the inscribed hexagon. Thus, a circumscribed hexagon is formed whose sides are the polars of the vertices of the inscribed hexagon, through the vertices of which they respectively pass (1002). Now, drawing the diagonals PM, NQ, OL, they are the polars of the intersections of the opposite sides of the inscribed hexagon, as PM, polar to the intersection of DE and CB (?); and hence they pass through a common point, as V. Thus we have Brianchon's theorem, viz.: *The lines joining the opposite angles of a circumscribed hexagon pass through a common point.*

1013.—The three following theorems are of frequent use in applying the theory of reciprocation.

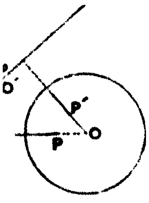


FIG. 514.

***1014. Theo.** — The angle included by two lines is equal to the angle included by the lines joining their poles to the centre of the auxiliary circle.*

DEM.—The pole of a line being in the perpendicular from the centre of the auxiliary circle upon the line (1000), O' is the supplement of O; hence $o = O$.

***Theo.**—The distances of any two points from the centre of the auxiliary circle are to each other as the distances of each point from the polar of the other.*

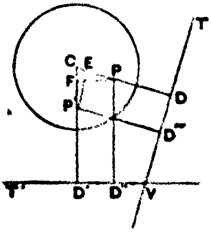


FIG. 515

DEM. P, P' being the points, and TV, T'V their polars respectively, we are to show that

$$CP : CP' :: PD'' : P'D'''.$$

By (*1001*) $R^2 = CP \times CD = CP' \times CD'$, R being the radius of the auxiliary circle. Whence $CP : CP' :: CD' : CD$. But $CP : CP' :: CF : CE$ (?), and there follows $CF : CE :: CD' : CD$, $CD' - CF$ $(= PD'')$

SCH.—This is known as *Salmon's Theorem*.

Theo.—*The anharmonic ratio of four points in a straight line is equal to that of the pencil formed by the four polars of these points.*

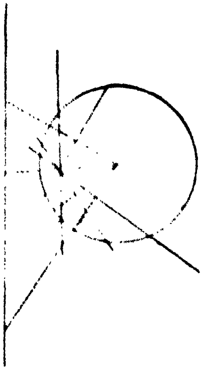


Fig. 516.

Dem.—1st. The polars pass through a common point and thus form a pencil (?). 2d. The angles included by the lines joining the four points with the centre, and those included by the polars are equal (?), hence the two pencils have the same anharmonic ratio.

SECTION VIII.

RADICAL AXES AND CENTRES OF SIMILITUDE OF CIRCLES.

1017. Def.—***The Power of a Point*** in the plane of a circle is the rectangle of the distances from the point to the intersections of the circumference by a line passing through the point.

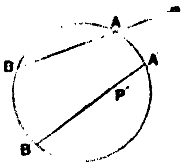


Fig. 517.

Ill.—Thus, the *power of a point* P, without the circle, is PA × PB ;—the power of a point within, as P', is P'A' × P'B' ; the power of a point in the circumference is *zero*, since one of the distances is then 0 ;—the power of the centre is the square of the radius.

¶. Cor.—*The power of a given point with respect to a given circle is a constant quantity.*

Thus PA × PB = the square of the tangent from P to the circle, in whatever position PB lies, so long as it passes through P. So also P'A' × P'B' = the rectangle of the segments of any other chord passing through P'.

Def.—***The Radical Axis*** of *Two Circles* is the locus of the point whose powers with respect to the two circles are equal.

1020. Prop.—*The Radical Axis of two circles is a right line.*

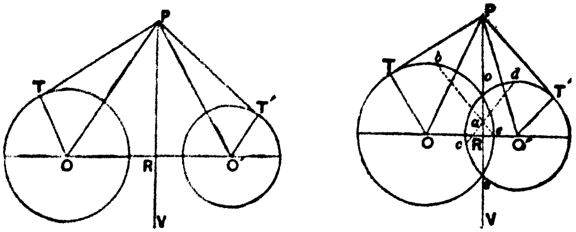


FIG. 518.

DEM.—Join the centres of the two circles O, O', and take a point R on this line such that $\overline{OR}^2 - \overline{O'R}^2 = \overline{OT}^2 - \overline{O'T'}^2$, or $\overline{OR}^2 - \overline{OT}^2 = \overline{O'R}^2 - \overline{O'T'}^2$, and erect *PR* perpendicular to OO'. Then P being *any* point in this perpendicular, $\overline{OP}^2 - \overline{OR}^2 = \overline{O'P}^2 - \overline{O'R}^2$. Adding this to the preceding equation, we have $\overline{OP}^2 - \overline{OT}^2 = \overline{O'P}^2 - \overline{O'T'}^2$, or $\overline{PT}^2 = \overline{PT'}^2$. ∴ PT = P'T', PT and PT' being tangents to the circles from any point in PR. Hence PV is the radical axis of the two circles.

1021. COR.—*When the circles are exterior to each other, the Radical Axis lies between them, touching neither; when they are tangent, either externally or internally, the radical axis is the common tangent; when they cut each other, the axis of the common chord produced.*

SCH.—When the circles intersect it might seem that the above demonstration fails for points within, as in the common chord. But, the powers of any point in this chord are still equal. Thus, at the intersections the powers are zero; and at any other point in the chord, as *a*, $ab \times ae = ad \times ac$, since each is equal to $ao \times as$.

1023. COR. *There is an infinite number of circles having their centres in the same right line, which have the same radical axis as any two given circles.*

Thus, in the first figure, PV being the radical axis of the circles O, O', letting circle O remain fixed, O' may vary indefinitely so that $\overline{O'R}^2 - \overline{O'T'}^2$ remains constant, and equal to $\overline{OR}^2 - \overline{OT}^2$.

1024. Prob.—*Given two circles, to draw their radical axis.*

SOLUTION.—Draw a common tangent, bisect it, and through the point of bisection draw a line perpendicular to the line joining the centres. When the circles are tangent to each other, the distance between the points of tangency is 0; hence the perpendicular is erected at this point. When they intersect, produce the common chord, or use the first method.

1025. **Prop.**—***When two circles cut each other orthogonally, that is, at right angles, the square of the radius of either is equal to the power of its centre with respect to the other.***

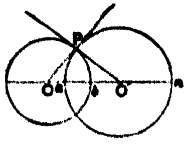


Fig. 519.

Ill.—The power of O with respect to circle O′ is $Oa \times On = \overline{OP}^2$, and of O′ with respect to circle O, $O'b \times O'm = \overline{O'P}^2$; since, as the circles cut each other orthogonally, their tangents are at right angles, and the tangent to either passes through the centre of the other.

1026. **Prop.**—*The radical axes of a system of three circles whose centres are not in the same straight line, intersect at a common point.*

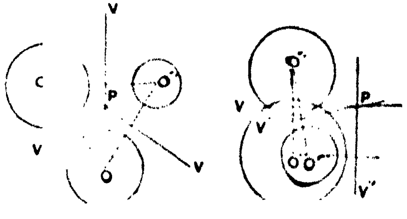


Fig. 520.

Dem.—Since O, O′, O″ are not in a straight line, the radical axes of O, O′, and O, O″, as PV″ and PV intersect. Let P be their common point. Now the power of P with respect to O′ is equal to its power with respect to O″, since each is equal to its power with respect to O. Hence P is a point in the radical axis of O′, O″.

1027. Cor.—*If the centres are in the same straight line, their radical axes are parallel, and the common point is at infinity.*

1028. Def.—The intersection of the radical axes of three circles is called their ***Radical Centre.***

CENTRES OF SIMILITUDE.

1029. Def.—If the line joining the centres of two circles be divided externally, as at C, and internally, as at C′, in the ratio of the radii, these points are respectively the *External* and the *Internal* ***Centres of Similitude*** of the two circles.

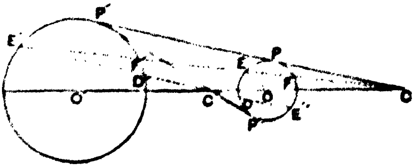


Fig. 521.

Ill.—If CO : CO′ :: EO : E′O′, C is the external centre of similitude; and, if C′O : C′O′ :: EO : E′O′, C′ is the internal centre of similitude.

The student should construct the figure when the circles are tangent externally,—when they are tangent internally,—and when one is wholly within the other.

QUERY.—How are the centres of similitude situated in the three different relative positions of the circles?

1030. Prop.—*In two circles the line passing through the extremities of two parallel radii on the same side of the line passing through the centres, intersects this line in the external centre of similitude, and if the radii are on opposite sides of this line the intersection is the internal centre of similitude.*

The proof consists in showing that the line passing through the centres is divided as above. Let the student show it for the three different positions of the circles.

1031. COR. 1.—Conversely, *If any transversal be drawn from either centre of similitude, the radii drawn to the intersections are parallel.*

Thus in the last figure, since CO : CO′ : EO : E′O′, and the triangles have the angle C common, EO and E′O′ are parallel.

1032. COR. 2.—*Tangents drawn at the alternate intersections of a transversal through the external centre of similitude are parallel; also, those at the mean intersections, and those at the extreme intersections, if the transversal be drawn through the internal centre of similitude.*

This follows as a consequence of the parallelism of the corresponding radii, to which the tangents are perpendicular. Thus, tangents at E and E′ are parallel, as are those at F and F′. So, also, tangents D and D′, and at E′ and E″ are parallel.

1033. DEF.—The extremities of two parallel radii on the same side of the line joining the centres are called *Homologous Points*, and those of non-parallel radii where the transversal cuts the circumferences, as E, F′, are called *Anti-Homologous Points.*

1034. COR. 2.—*The distances of a centre of similitude from two homologous points are to each other as the radii.*

1035. COR. 3.—*The centres of similitude and the centres of the circles are four harmonic points.*

1036. Prop.—*If a circle touch two others, the line joining their points of contact passes through the external centre of similitude of*

the latter if the contacts are both external or both internal; ...... through the internal centre of similitude if the contacts are the one external and the other internal.

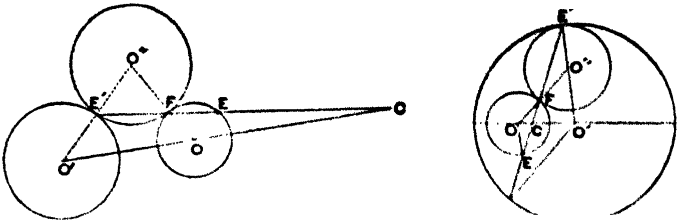


Fig 592.

DEM.—In either case let O″ be the circle tangent to O, and O′; and through the points of tangency draw E′C. The angle O″E′F = O″FE′ = EFO = FEO; whence OE and O′E′ are parallel, and the similar triangles CEO, CE′O′ give CO : CO′ : : OE : O′E′. [The student should make the other constructions.]

1037. Prob.—*To draw a line parallel to a given line so that the distance between the extreme intersections with two given circles shall be a maximum.*

SOLUTION.—Draw a line through the internal centre of similitude and parallel to the given line. Now, at the extreme intersections draw tangents, and it will become evident that the line first drawn is a maximum. [The student should make the figure and fill out the proof.]

If the circles are wholly exterior to each other, the distance between the mean intersections is a minimum.

9. CONCLUDING NOTE.—Our limits preclude our pursuing these topics farther. We have given enough to make the language of the Modern Geometry intelligible, and to afford some insight into its character. One of the best elementary resources for the English student who wishes to pursue the subject at greater length, is MULCAHY'S *Principles of Modern Geometry*, Dublin, 1862. It is, however, much to be regretted, that there is no English treatise which presents the elements of this subject with the philosophic elegance of the French. The best of the latter is ROUCHÉ and COMBEROUSSE'S Treatise on Elementary Geometry. For a more extended view of the subject, SALMON or WHITWORTH will furnish the English student; but he who would be proficient must read the works of CHASLES and PONCELET, who are the great authorities.

OLNEY'S MATHEMATICAL SERIES.

ELEMENTS

OF

TRIGONOMETRY,

PLANE AND SPHERICAL.

By EDWARD OLNEY,

PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF MICHIGAN.

NEW YORK:
SHELDON & COMPANY,
No. 8 MURRAY STREET.
1876.

Stoddard's Mathematical Series.

STODDARD'S JUVENILE MENTAL ARITHMETIC . .	$ 25
STODDARD'S INTELLECTUAL ARITHMETIC	40
STODDARD'S RUDIMENTS OF ARITHMETIC . . .	50
STODDARD'S NEW PRACTICAL ARITHMETIC	1 00

SHORT AND FULL COURSE FOR GRADED SCHOOLS.

STODDARD'S PICTORIAL PRIMARY ARITHMETIC . .	30
STODDARD'S COMBINATION ARITHMETIC . . .	75
STODDARD'S COMPLETE ARITHMETIC	1 25

The Combination School Arithmetic being Mental and Written Arithmetic in one book, will alone serve for District Schools. For Academies a full high course is obtained by the Complete Arithmetic and Intellectual Arithmetic.

OLNEY'S HIGHER MATHEMATICS.

A COMPLETE SCHOOL ALGEBRA in one vol., 390 pages, $1.50. Designed for Elementary and higher classes in Schools and Academies. By Prof. Edward Olney, University of Michigan.

A GEOMETRY AND TRIGONOMETRY in one vol. By Prof. Edward Olney. One vol. 8vo, ... Price $ 2.50.

A GENERAL GEOMETRY AND CALCULUS in one vol. . . 2.50.

The other books of Olney's Series will be published as rapidly as possible.

Entered according to Act of Congress, in the year 1870,
By SHELDON & COMPANY
In the Office of the Librarian of Congress at Washington.

CONTENTS.

PART IV.—TRIGONOMETRY.

CHAPTER I.

PLANE TRIGONOMETRY.

SECTION I.

DEFINITIONS AND FUNDAMENTAL RELATIONS BETWEEN THE TRIGONOMETRICAL FUNCTIONS OF AN ANGLE (OR ARC).

PAGE.

Definitions 1–5
Fundamental Relations of the Trigonometrical Functions of an Angle 5–7
Signs of the Functions 7–10
Limiting Values of the Functions 10–14
Functions of Negative Arcs 15–16
Circular Functions 16
Exercises 17–20

SECTION II.

RELATIONS BETWEEN THE TRIGONOMETRICAL FUNCTIONS OF DIFFERENT ANGLES (OR ARCS).

Functions of the Sum or Difference 20–25
Functions of Double and Half Angles 26
Exercises 26–29

SECTION III.

FORMULÆ for rendering Calculable by Logarithms the Algebraic Sum of Functions 29–30
Exercises 30–31

SECTION IV.

CONSTRUCTION AND USE OF TABLES.

Definitions 31–32
To Compute a Table of Natural Functions 32–33
To Compute a Table of Logarithmic Functions 33–34
Exercises in the Use of the Tables 34–39
Functions of Angles near the Limits of the Quadrant 39–42
Exercises 43

SECTION V.

SOLUTION OF PLANE TRIANGLES.

PAGE.

Of Right Angled Triangles 44–45
Exercises and Examples 45–48
Practical Applications 48–49
Of Oblique Angled Plane Triangles 49–51
Exercises 51–55
Oblique Triangles Solved by means of Right Angled Triangles 55–56
Exercises 56–57
Functions of the Angles in Terms of the Sides 58–59
Exercises 59–60
Area of Plane Triangles 60–61
Practical Applications 61–64

CHAPTER II.

SPHERICAL TRIGONOMETRY.

INTRODUCTION.

PROJECTION OF SPHERICAL TRIANGLES.

Definitions and Fundamental Propositions 65–66
Projection of Right Angled Spherical Triangles 66–71
Projection of Oblique Angled Spherical Triangles 71–73

SECTION I.

SOLUTION OF RIGHT ANGLED SPHERICAL TRIANGLES.

Definitions 73–75
Exercises 75–76
Napier's Rules 76–78
Determination of Species 79–81
Exercises in Solution of Right Angled Spherical Triangles 81–84
Quadrantal Triangles 84–85

SECTION II.

OF OBLIQUE ANGLED SPHERICAL TRIANGLES.

To find the Segments of a Side made by a Perpendicular let fall from the opposite angle 85–86
The Relation of the Sides and Opposite Angles 87
Solution of Oblique Spherical Triangles by Napier's Rules, in Three Problems 87–90
Exercises

SECTION III.

GENERAL FORMULÆ.

PAGE.
Angles as Functions of Sides, and Sides as Functions of Angles. 96-100
Gauss's Equations 100-101
Napier's Analogies 102
Exercises in the Use of these Formulæ 102-107

SECTION IV.

Area of Spherical Triangles 108-110
Practical Applications of Spherical Trigonometry 110-113

TABLES.

Introduction to the Table of Logarithms 1-10
Table of Logarithms of Numbers 11-28
Table of Natural Sines and Cosines, and of Logarithmic Sines, Cosines, Tangents, and Cotangents 29-74
Table for Precise Calculation of Functions near their Limits 75-78
Table of Tangents and Cotangents 79-88

PART IV.

TRIGONOMETRY.

CHAPTER I.

PLANE TRIGONOMETRY.

SECTION I.

DEFINITIONS AND FUNDAMENTAL RELATIONS BETWEEN THE TRIGONOMETRICAL FUNCTIONS OF AN ANGLE (OR ARC).

1. Trigonometry is a part of Geometry which has for its subject-matter, *Angles.* It is chiefly occupied in presenting a scheme for measuring and comparing angles, by means of certain auxiliary lines called *Trigonometrical Functions*, in investigating the relations between these functions, and in the solution of triangles by means of the relations between their sides and the trigonometrical functions of their angles.

2. Plane Trigonometry treats of plane angles and triangles, in distinction from *Spherical Trigonometry*, which treats of spherical angles and triangles.

3. A Function is a quantity, or a mathematical expression, conceived as depending upon some other quantity or quantities for its value.

ILL'S.—A man's wages *for a given time* is a function of the amount received per day; or, in general, his wages is a function of *both* the *time* of service and the *amount* received per day. Again, in the expressions $y = 2ax^3$, $y = x^2 - 2bx + 5$, $y = 2\log ax$, $y = a^x$, y is a function of x; since, the numbers 2, 5, a and b being considered fixed or constant, the value of y depends upon the value we assign to x. For a like reason such expressions as $\sqrt{a^2 - x^2}$, and $3ax^2 - 2\sqrt{x}$, may be spoken of as functions of x. Once more, the area of a triangle is a function of its base and altitude.

4. Angles as Functions of Arcs.—We have learned in Geometry (PART II., SEC. VI.), that angles and arcs may be treated as functions of each other; and that, if the angles be taken at the centre of the same or equal circles, the arcs intercepted have the same ratio as the angles themselves, and hence may be taken as their measures or representatives. For trigonometrical purposes, an angle is considered as measured by an arc struck with a radius 1, from the angular point as a centre.

5. ***A Degree*** being the $\frac{1}{360}$ part of the circumference of a circle, becomes the measure of $\frac{1}{90}$ of a right angle; and, for convenience, it is customary to speak of such an angle as *an angle of one degree,* of four times as large an angle as an angle of four degrees, etc., applying the term directly to the angle. A small circle written at the right and a little above a number indicates degrees (°).

6. ***A Minute*** is $\frac{1}{60}$ part of a degree. Minutes are designated by an accent ('). ***A Second*** is $\frac{1}{60}$ part of a minute. Seconds are indicated by a double accent ("). Smaller divisions of angles (or arcs) are most conveniently represented as decimals of a second, though the designations *thirds, fourths,* etc., are sometimes met with, and signify further subdivisions into 60ths. 5° 12' 16" 13''' is read, "5 degrees, 12 minutes, 16 seconds, and 13 thirds."

ILL'S.—In *Fig.* 1 AOB is an angle of 35°, because the measuring arc *ab* contains 35 of the 360 equal parts into which the circumference whose radius is *Oa*, could be divided. In like manner BOC is an angle of 7°. BOC = $\frac{1}{5}$AOB = $\frac{1}{6}$AOC. Hence, it becomes evident that we may use the numbers 35, 7, and 42, to represent the respective angles AOB, BOC, and AOC, or the corresponding arcs *ab*, *bc*, and *ac*

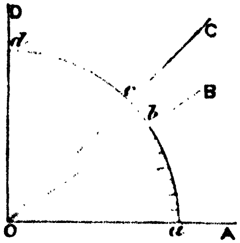


FIG. 1.

7. ***A Quadrant*** is an arc of 90°, and is the measure of a right angle; hence, a right angle is called an angle of 90°. Thus arc *ad, Fig.* 1, = 90°, or angle AOD = 90°.

8. ***The Complement*** of an angle or arc is what remains after subtracting the angle or arc from 90°. The *Supplement* of an angle or arc is what remains after subtracting the angle or arc from 180°.

ILL'S.—In *Fig.* 1, the angle BOD is the complement of AOB, and the arc *bd* is the complement of arc *ab*. The complement of 35° is 90° − 35° = 55°. The supplement of 35° is 180° − 35° = 145°.

9. A Quadrant is often represented by $\frac{1}{2}\pi$, since π is the semi-circumference when the radius is unity. When this notation is used, the ***Unit Arc*** becomes $\frac{180^\circ}{\pi} = 57^\circ.29578$ nearly, or $57^\circ\ 17'\ 44''.8$ +, which is an arc equal in length to the radius.

10. For trigonometrical purposes, an angle is conceived as generated by the revolution of a line about the angular point, and hence *may have any value whatever*, not only from 0° to 180°, but from 0° to 360°, and even to any number of degrees greater than 360°, as 1280°, etc. An angle of 45° is generated by ⅛ of a revolution, 90° by ¼ of a revolution, 180° by ½ a revolution, 270° by ¾ of a revolution, 360° by one revolution, 450° by 1¼ revolutions, 1280° by 3$\frac{5}{9}$ revolutions, etc., etc.

11. In accordance with the conception of an angle as generated by a revolving line, the measuring arc is considered as *originating* at the first position of the revolving line (*i. e.*, with one side of the angle), and terminating in the line after it has generated the angle under consideration (*i. e.*, with the other side of the angle). The first extremity is called the ***Origin*** of the arc, and the other the ***Termination.***

ILL'S.—In *Fig.* 1, let the angle AOB be considered as generated by a line starting from the position OA, and revolving around the point O, from right to left,* till it reaches the position OB. O*a* being taken as unity, the arc *ab* is the measuring arc of the angle AOB; *a* is its *origin*, and *b* its *termination*.

12. In the generation of angles by means of a revolving line, the normal motion is considered to be from right to left, and the quadrants are numbered 1st, 2d, 3d, and 4th, in the order in which they are generated.

13. The Trigonometrical Functions are eight in number; viz., *sine, cosine, tangent, cotangent, secant, cosecant, versed-sine*, and *coversed-sine*. These lines are functions of angles, or, what amounts to the same thing, of arcs considered as measures of angles, and are the characteristic quantities of trigonometry.

14. The Sine of an angle (or arc) is a perpendicular let fall from the termination of the measuring arc upon the diameter passing through the origin of the arc. Thus in *Fig.* 2, *bd* is in each case the sine of the angle AOB, or of the arc *axb*.

---

* The pupil will understand that, if he imagines himself standing at the centre of motion, as the moving body or point passes before him, the distinctions "from right to left," and "from left to right," are easily made.

15. The Trigonometrical Tangent of an angle (or arc) is a tangent drawn to the measuring arc at its origin, and limited by the produced diameter passing through the termination of the arc. Thus in *Fig.* 2, *ac* is in each case the tangent of the angle AOB, or of the arc *axb*.

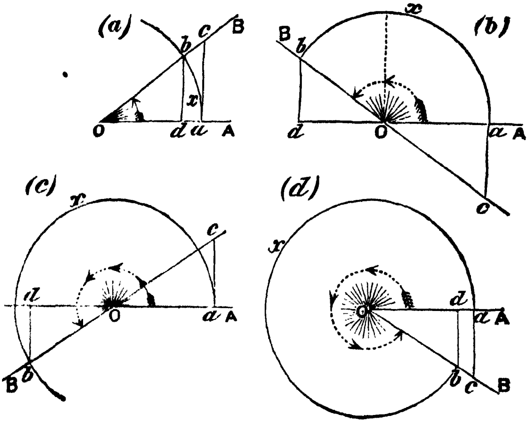


FIG. 2.

16. The Secant of an angle (or arc) is the distance from the angular point, or centre of the measuring circle, to the extremity of the tangent of the same angle (or arc). Thus in *Fig.* 2, O*c* is in each case the secant of the angle AOB, or of the arc *axb*.

17. The Versed-Sine of an angle (or arc) is the distance from the foot of the sine of the same angle (or arc) to the origin of the measuring arc. Thus, in *Fig.* 2, *da* is in each case the versed-sine of the angle AOB, or of the arc *axb*.

18. The prefix *co*, in the names of the four trigonometrical functions in which it occurs, is an abbreviation for the word *complement*. Thus cosine means complement-sine, *i. e.*, the sine of the complement; cotangent means tangent of the complement; etc. The cosine of 40° is the sine of 90° − 40°, or 50°; the cosine of 110° is the sine of 90° − 110°, or − 20°; the cotangent of 30° is the tangent of 60°; the cosecant of 200° is the secant of − 110°.

19. Construction of the Complementary Functions. —Let us now see how the complementary functions are constructed with reference to their primitives, premising that all arcs in *Fig.* 3, reckoned from A, are to be reckoned around from right to left in this discussion. 1st. Let AP be any arc less than 90° ; then 90° − AP = *a*P is its complement. Now considering *a* as the origin and P the termination of this complementary arc, P*d* is its sine, *at* its tangent, O*t* its secant, and *ad* its versed-sine. Hence, P*d*, *at*, O*t*, and *ad* are respectively the cosine, cotangent, cosecant, and coversed-sine of the arc AP, or the angle AOP. 2d. Letting APP′ be any arc between 90° and 180°, its complement is 90° − APP′ or − *a*P′, the − sign signifying that the arc is reckoned backward from P′ to *a*. But as the *values* of the functions will be the same whether the origin be taken at P′ or at *a*, we may take *a* as the origin of this complementary arc, and P′ as its termination, whence P′*d*′ becomes its sine, *at*′ its tangent, O*t*′ its secant, and *ad*′ its versed-sine. Therefore P′*d*′, *at*′, O*t*′, and *ad*′, are respectively the cosine, cotangent, cosecant, and coversed-sine of the arc APP′, or the angle AOP′. 3d. In like manner, *a*P″ is shown to be the complement of arc APP′P″; and as P″*d*″, *at*, O*t*, and *ad*″ are respectively the sine, tangent, secant, and versed-sine of this complement, they are the corresponding cofunctions of the arc APP′P″, or the salient angle AOP″. 4th. In the same way, it appears that P‴*d*‴, *at*′, O*t*′, and *ad*‴ are the cosine, cotangent, cosecant, and coversed-sine of the arc APP′P″P‴, or the salient angle AOP‴. *Observe that* a, *a point on the measuring arc* 90° *from the primitive origin, is the origin of all the complementary functions.*

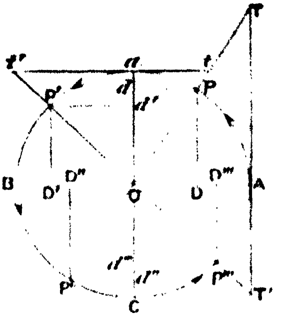


Fig. 3.

SCH.—It will readily appear from the figure that the cosine of an angle (or arc) is always equal to *the distance from the foot of the sine to the vertex of the angle* (or the centre of the measuring arc). This is the more convenient practical definition. Thus the cosine of AP is P*d* = DO; the cosine of APP′ is P′*d*′ = D′O, etc.

20. *Notation.*—Letting x represent any angle (or arc), the several trigonometrical functions of it are written $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$, $\operatorname{vers} x$, and $\operatorname{covers} x$. They are read "sine x," "cosine x," "tangent x," "cotangent x," etc.

FUNDAMENTAL RELATIONS BETWEEN THE TRIGONOMETRICAL FUNCTIONS OF AN ANGLE (OR ARC).

[*Note.*—These fundamental relations must be made perfectly familiar. They must be memorized, and be as familiar as the Multiplication Table. The student can do nothing in Trigonometry without them.]

☞ The discussions in this treatise all proceed upon one general plan; viz.,—***First obtain the particular property of the***

sine and cosine, and from this deduce all the others according to the dependencies shown in the following proposition.

21. *Prop.*—*The Fundamental Relations which the Trigonometrical Functions sustain to each other are:*

(1) $\sin^2 x + \cos^2 x = 1$;

(2) $\tan x = \dfrac{\sin x}{\cos x}$;

(3) $\cot x = \dfrac{\cos x}{\sin x}$;

(4) $\cot x = \dfrac{1}{\tan x}$;

(5) $\sec x = \dfrac{1}{\cos x}$;

(6) $\operatorname{cosec} x = \dfrac{1}{\sin x}$;

(7) $\sec^2 x = 1 + \tan^2 x$;

(8) $\operatorname{cosec}^2 x = 1 + \cot^2 x$;

(9) $\operatorname{vers} x = 1 - \cos x$;

(10) $\operatorname{covers} x = 1 - \sin x$.

(The forms $\sin^2 x$, $\sec^2 x$, etc., signify the square of the sine, the square of the secant of x, etc., and are read "sine square x," "secant square x," etc. The student should distinguish between $\sin^2 x$, and $\sin x^2$.)

Dem.—In *Fig.* 4, let x represent any arc as **AP**, less than 90°. Then $\mathsf{PD} = \sin x$, **OD** or $\mathsf{P}d = \cos x$, $\mathsf{AT} = \tan x$, $\mathsf{OT} = \sec x$, $at = \cot x$, $\mathsf{O}t = \operatorname{cosec} x$, $\mathsf{AD} = \operatorname{versin} x$, and $ad = \operatorname{coversin} x$.

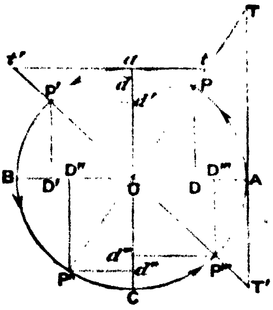


FIG. 4.

(1). In the right-angled triangle **POD**, $\overline{\mathsf{PD}}^2 + \overline{\mathsf{OD}}^2 = \overline{\mathsf{OP}}^2$, or $\sin^2 x + \cos^2 x = 1$, since $\mathsf{OP} = \text{radius} = 1$.

(2). From the similar triangles **POD** and **TOA**,

$$\frac{\mathsf{AT}}{\mathsf{OA}} = \frac{\mathsf{PD}}{\mathsf{OD}}, \text{ or } \tan x = \frac{\sin x}{\cos x}.$$

(3). From the similar triangles **PO***d* and *t***O***a*,

$$\frac{at}{\mathsf{O}a} = \frac{\mathsf{P}d}{\mathsf{O}d}, \text{ or } \cot x = \frac{\cos x}{\sin x}.$$

(4). Multiplying (2) and (3) together,

$$\tan x \cot x = \frac{\sin x \cos x}{\cos x \sin x} = 1, \text{ or } \tan x = \frac{1}{\cot x}$$

(5). From the similar triangles **OTA** and **OPD**,

$$\frac{\mathsf{OT}}{\mathsf{OA}} = \frac{\mathsf{OP}}{\mathsf{OD}}; \text{ but OP and OA each } = 1, \therefore \sec x = \frac{1}{\cos x}.$$

(6). From the similar triangles O*ta* and OP*d*,

$$\frac{Ot}{Oa} = \frac{OP}{Od}, \text{ or } \operatorname{cosec} x = \frac{1}{\sin x}.$$

(7). From the right-angled triangle OAT,

$$\overline{OT}^2 = \overline{OA}^2 + \overline{AT}^2, \text{ or } \sec^2 x = 1 + \tan^2 x.$$

(8). From the right-angled triangle O*at*,

$$\overline{Ot}^2 = \overline{Oa}^2 + \overline{at}^2, \text{ or } \operatorname{cosec}^2 x = 1 + \cot^2 x.$$

(9). $AD = AO - OD$, or $\operatorname{vers} x = 1 - \cos x$.

(10). $ad = aO - Od$, or $\operatorname{covers} x = 1 - \sin x$.

Thus the fundamental relations of the functions are established for an arc less than 90°. But it will readily appear that the relations are the same for any other arc. For example, let $x = AP'$ be any arc between 90° and 180°. Then the triangle P′D′O gives $\sin^2 x + \cos^2 x = 1$, since $P'D' = \sin x$, and $OD' = \cos x$. The similar triangles P′D′O and OAT′ give $\frac{AT'}{OA} = \frac{P'D'}{D'O}$, or $\tan x = \frac{\sin x}{\cos x}$; and the similar triangles P′*d*′O and *t*′*a*O give $\cot x = \frac{\cos x}{\sin x}$. In like manner let the student observe the relations when $x = APP'P''$, or an arc between 180° and 270°. So also when $x = APP'P''P'''$, or an arc between 270° and 360°.

22. Cor. 1.—*The tangent and cotangent of the same angle are reciprocals of each other; so also are the secant and cosine, and the cosecant and the sine.* Thus, if $\tan x = 3$, $\cot x = \frac{1}{3}$; since $\cot x = \frac{1}{\tan x}$. If $\sec x = 2$, $\cos x = \frac{1}{2}$; since $\sec x = \frac{1}{\cos x}$, or $\cos x = \frac{1}{\sec x}$.

23. Cor. 2.—*Sines and cosines cannot exceed* 1. *Tangents and cotangents can have any values from* 0 *to* ∞. *Secants and cosecants can have any values between* 1 *and* ∞. *Versed-sines and coversed-sines can have any values between* 0 *and* 2. These conclusions will readily appear from the definitions, and an inspection of *Fig.* 4.

SIGNS OF THE TRIGONOMETRICAL FUNCTIONS.

24. Prop.—*Angles* (*or arcs*) *considered as generated from right to left being called positive* * *and marked* +, *those considered as generated from left to right are to be called negative and marked* −.

---

* This is purely an arbitrary convention. We might equally well reverse it

Dem.—This is a direct application of the significance of the + and − signs. (See Complete School Algebra, pp. 20–23.) Thus, in *Fig.* 5, if the angle **AOP**, considered as generated by the revolution of a line from the position **OA** in the direction of the arrow-head (from right to left), is called positive and marked +, an angle generated by the motion of a line from the position **OA** in the *opposite* direction (from left to right), as the angle **AOP″** thus generated, is to be considered negative and marked −. Let it be carefully observed that it is the assumed *direction* of the motion of the generatrix that determines *the sign of the angle (or arc). Two lines meeting at a common point may be considered as designating either a + or a − angle, according to the direction of motion assumed.* Thus the lines **OA** and **OP′**, *Fig.* 5, may form the positive angle measured by the arc **APP′**, or the negative, salient angle measured by the negative arc **AP‴P″P′**. Q. E. D.

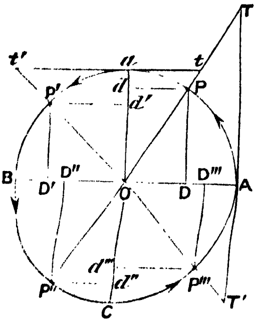


Fig. 5.

25. Prop.—*Radius being considered as always extending in the same direction, viz., from the centre toward the circumference, is always positive.*

26. Prop.—*The sign of the sine of an angle between* 0° *and* 180° *being* +, *that of an angle between* 180° *and* 360° *is* −.

Dem.—In *Fig.* 5, we observe that the sines of all angles terminating in the 1st and 2d quadrants, *i. e.*, between 0° and 180°, may be considered as measured from the primary diameter **AB**, *upward*, while those of angles terminating in the 3d and 4th quadrants, *i. e.*, between 180° and 360°, are reckoned downward from the same line; hence, the former being called +, the latter should be −, *as the two species are estimated in opposite directions.* Q. E. D.

A more elegant conception is to consider the sine as projected upon the diameter vertical to that passing through the origin, as *a***C**; whence **O***d* is the sine of **AOP** (or arc **AP**). Now this line evidently is 0 when the angle is 0; and as the angle increases, the sine increases, being generated from **O** *upward*, and hence is called +. This is the same conception as we use in the case of the cosine. Adopting it, we see that sines reckoned from **O** upward are +, and downward −. Cosines reckoned from **O** to the *right*, are +, and to the left, −.

27. Cor.—*The cosecant of an arc has the same sign as its sine,* since $\operatorname{cosec} x = \frac{1}{\sin x}$; and as 1, being the radius, is +, the sign of $\frac{1}{\sin x}$ is the same as the sign of $\sin x$.

28. Prop.*—The sign of the cosine of an angle between 0° and 90°, and between 270° and 360°, is +, while that of an angle between 90° and 270° is —.*

Dem.—In *Fig.* 5, we observe that the cosines of all angles terminating in the 1st and 4th quadrants, may be considered as estimated from the centre toward the right, as OD, OD‴; while correspondingly, the cosines of angles terminating in the 2d and 3d quadrants will be estimated from the centre toward the left, as OD′, OD″. Hence, by reason of this opposition of direction, the former are called +, and the latter —. Q. E. D.

29. Cor.—*The secant of an angle has the same sign as its cosine,* since these functions are reciprocals of each other. (See ***27.***)

---

30. Prop.*—The sign of the tangent of an angle between 0° and 90°, and also between 180° and 270°, is +; while that of an angle between 90° and 180°, and between 270° and 360°, is —.*

Dem.—Since $\tan x = \frac{\sin x}{\cos x}$, when $\sin x$ and $\cos x$ have like signs, $\tan x$ is +, by the rules of division; and when $\sin x$ and $\cos x$ have different signs, $\tan x$ is —. Now, in the 1st and 3d quadrants* the signs of $\sin x$ and $\cos x$ are alike, hence in these quadrants $\tan x$ is plus; but in 2d and 4th quadrants $\sin x$ and $\cos x$ have unlike signs, and consequently in these $\tan x$ is —. Q. E. D.

31. Cor.—*The sign of the cotangent is the same as the sign of the tangent of the same angle,* since $\cot x = \frac{1}{\tan x}$.

---

32. Prop.*—Versed-sine and coversed-sine are always +.*

Dem.—$\text{Vers}\, x = 1 - \cos x$; and as $\cos x$ cannot exceed 1, $1 - \cos x$ is always +. In like manner, $\text{covers}\, x = 1 - \sin x$; and as $\sin x$ cannot exceed 1, $1 - \sin x$ is always +. Q. E. D.

Sch. 1.—It is essential that the law of the signs, as explained above, be well understood, and the facts fixed in memory. *Fig* 6 will aid the student in fixing the law in the memory. Having this constantly before the mind, and remembering that tan and cot are + when sin and cos have like signs, and — when they have unlike, and that cos and sec have like signs, as also sin and cosec, or, more simply, that $\tan = \frac{\sin}{\cos}$, $\cot = \frac{1}{\tan}$, $\sec = \frac{1}{\cos}$, and $\text{cosec} = \frac{1}{\sin}$, the student cannot fail to know the sign of a function at a glance.

It will be of service to remember that versed-sine and coversed-sine, and *all* the functions of angles of the 1st quadrant, are +; but that of the other functions than the versed-sine and coversed-sine, of angles terminating in the other quadrants, but *two* are + in each quadrant.

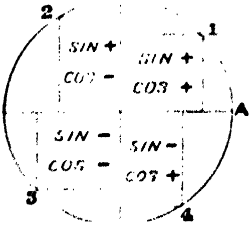


Fig. 6.

---

* This is a convenient elliptical form for "an angle whose measuring arc terminates in the 1st quadrant," etc.

SCH. 2.—The signs of functions of angles greater than 360° are readily determined by observing in what quadrant the measuring arc terminates. Thus, sin 570° is —, since an arc of 570° terminates in the 3d quadrant. In any given case, the sign of the function is the same as the sign of the same function of the remainder left after dividing the arc by 360°, or 2π. Thus tan 1180° is the same as tan 100°; *i. e.*, it is —. The same may also be said of the *value* of the function.

LIMITING VALUES OF THE TRIGONOMETRICAL FUNCTIONS.

33. ***Prop.***—*Sin* $0° = \mp 0$, *sin* $90° = 1$, *sin* $180° = \pm 0$, *sin* $270° = -1$, *sin* $360° = \mp 0$, *and the limits within which the sines of all angles are comprised, are* $+1$ *and* -1.

DEM.—Let the point P be supposed to start from A and move around the circumference from right to left, and let x represent the angle (or arc) generated. Now, when P is at A, $x = 0$, and $PD = 0$. Moreover, if we consider the sine P‴ D‴ as reaching its limit, by the moving of P‴ from some point in the fourth quadrant to the origin, the sine of 0° becomes — 0, since *what is true of a varying quantity all the way as it approaches its limit, is assumed true at the limit.* But, if the sine reaches its limit by the passage of the point P from some point in the first quadrant to the origin, the sine of 0° is to be considered as +, since the function is + as it approaches its limit. ∴ sin 0° = ∓ 0.* As x increases from 0° to 90° (*i. e.*, as P passes from A to a), PD is + and increases from 0 to 1 (+ 0 to + 1). ∴ sine 90° = 1. As x continues to increase from 90°, sin x diminishes and becomes 0 at 180°. To determine the sign of sin 180°, we notice that it is + when the point P approaches this limit from the second quadrant, and — when it approaches it from the third quadrant, ∴ sin 180° = ± 0. Conceiving x to go on increasing from 180°, sin x appears *below* AB and is —, and beginning at — 0 *diminishes* (a *numerical increase* of a negative quantity being considered an absolute decrease) till at 270° it becomes — 1. ∴ sin 270° = — 1. As x passes from 270° to 360°, sin x *increases* (see above) from — 1 to 0. The *sign* of this limit is ambiguous, as appears by regarding the limit as reached by the angle passing from something less than 360° to 360°, whence we have —, and also as reached by the angle passing *back* from something *greater* than 360° to 360°, whence the sign is +. ∴ sin 360° = ∓ 0. Finally, as it is evident that these values would recur in the same order as the point P passed around again, the above comprise all real values of sines of angles. Q. E. D.

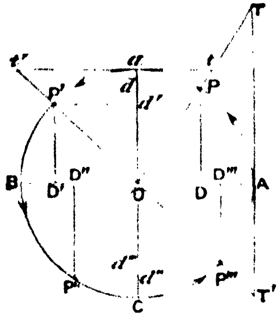


FIG. 7.

---

* It has been customary to disregard the sign of 0, in such a series as — ∞ . . . — 3, — 2, —1, ∓ 0, + 1, + 2, + 3, ... + ∞, whereas it
above.

34. Prop.—*Cos* $0° = 1$, *cos* $90° = \pm 0$, *cos* $180° = -1$, *cos* $270° = \mp 0$, *cos* $360° = 1$, *and the cosines of all angles are comprised between* $+1$ *and* -1.

Dem.—Using the same figure and the same conception as in the last demonstration, it is evident that as **P** approaches **A**, *from either direction*, that is as x approaches 0°, the cosine **OD** approaches to equality with the radius and reaches it at $x = 0$, being $+$ in either case. $\therefore \cos 0 = 1$. As x approaches 90° from a less value, *i. e.*, from the first quadrant, $\cos x$ is $+$, and approaching $+0$; but as x approaches 90° from some greater value, *i. e.*, from the second quadrant, $\cos x$ is $-$ and approaching -0. $\therefore \cos 90° = \pm 0$. In like manner we observe that as x increases from 90° to 180°, $\cos x$ *decreases* (see Dem. of 33) from 0 to -1, which it reaches at 180°, and this, whether the point is reached in one way or the other. $\therefore \cos 180° = -1$. Again, $\cos 270° = \mp 0$; since it is -0 if x passes to 270° from some value *less* than 270°, and $+0$, if it passes from some value *greater* than 270. While x passes through the fourth quadrant, $\cos x$ passes from 0 to $+1$, reaching the latter value when $x = 360°$. $\therefore \cos 360° = 1$. Finally, as it is evident that the above values would recur in the same order as the generating point passed around again, this discussion comprises all real values of a cosine.

35. Prop.—*Tan* $0° = \mp 0$, *tan* $90° = \pm \infty$, *tan* $180° = \mp 0$, *tan* $270° = \pm \infty$, *tan* $360° = \mp 0$, *and the tangents of all angles are comprised between* $+\infty$ *and* $-\infty$.

Dem. — $\text{Tan } 0° = \dfrac{\sin 0°}{\cos 0°} = \dfrac{\mp 0}{1} = \mp 0$. $\text{Tan } 90° = \dfrac{\sin 90°}{\cos 90°} = \dfrac{1}{\pm 0} = \pm \infty$ $\text{Tan } 180° = \dfrac{\sin 180°}{\cos 180°} = \dfrac{\pm 0}{-1} = \mp 0$. $\text{Tan } 270° = \dfrac{\sin 270°}{\cos 270°} = \dfrac{-1}{\mp 0} = \pm \infty$. $\text{Tan } 360° = \dfrac{\sin 360°}{\cos 360°} = \dfrac{\mp 0}{1} = \mp 0$. From these results it appears that the tangent may have any value whatever from $+\infty$ to $-\infty$; and as subsequent revolutions of the generatrix would evidently only repeat these values, these are all the real values of a tangent. (Moreover, all real values are comprised between $+\infty$ and $-\infty$). Q. E. D.

It is easy to observe the direction of the change in the tangent (whether it is increasing or decreasing) by observing the fraction $\dfrac{\sin}{\cos}$. As the arc increases in the first quadrant, the sine increases and the cosine decreases, for both of which reasons the tangent increases, and hence changes more rapidly than either sine or cosine. The student should observe the change in each of the four quadrants in the same manner.

Sch.—These values of the tangent are illustrated by *Fig.* 7. Thus **AT**, which is $+$, becomes $+0$ when **P** returns to **A**, or $x = 0$. Also **AT′**, which is $-$, and may be considered as the tangent of **AP‴**, a negative arc, becomes -0, when $\text{AP}''' = 0$. Again, if **AP** passes to **A*a***, **AT** passes to $+\infty$. But, if **P′** passes *back* to *a*, so that **A*a*P′** passes to **A*a***, or 90°, **AT′** passes to $-\infty$. $\therefore$ We see that tan 0 may be considered ∓ 0, and $\tan 90° = \pm\infty$. In like manner the other limits are illustrated.

37. Prop.—*Sec* $0° = 1$, *sec* $90° = \pm \infty$, *sec* $180° = -1$, *sec* $270° = \mp \infty$, *sec* $360° = 1$, *and all real values of this function are comprised between* 1 *and* $\pm \infty$, *and* -1 *and* $\mp \infty$.

Dem.—These values are the reciprocals of the corresponding values of the cosine. The student should obtain them from the cosine, and illustrate them from *Fig.* 7, observing the law of change. Thus beginning at $OA = 1$, which is the secant of 0°, the secant increases till $x = 90°$, when the secant OT becomes $+\infty$, if we consider this limit as reached thus; but $-\infty$, if we consider such an arc as **A*a*P**, whose secant is $-$ OT′, which becomes $-\infty$, when the point **P′** passes back to *a*.

Sch.—The series which represent the values of secants are, for the first and second quadrants, $+1, +2, +3, + \dots + m, \pm \infty, -m, - \dots -3, -2, -1$; for the third and fourth quadrants, $-1, -2, -3, - \dots - m, \mp \infty, +m, + \dots +3, +2, +1$, understanding the numbers in these series as representing a few terms of series which have an infinite number of terms of all values extending, in the first case, from $+1$ to $\pm \infty$, and thence to -1. It will be a good exercise for the pupil to write the series representing the values of each of the trigonometrical functions. Thus, for the sine, we have $\mp 0, +\frac{1}{4}, +\frac{1}{2}, +\frac{3}{4}, +1, +\frac{3}{4}, +\frac{1}{2}, +\frac{1}{4}, \pm 0$, for the first and second quadrants, understanding that all values intermediate between those represented are included. For the second and third quadrants, we have, $+1, +\frac{3}{4}, +\frac{1}{2}, +\frac{1}{4}, \pm 0, -\frac{1}{4}, -\frac{1}{2}, -\frac{3}{4}, -1$.

38. Prop.—*Cosec* $0° = \mp \infty$, *cosec* $90° = 1$, *cosec* $180° = \pm \infty$, *cosec* $270° = -1$, *cosec* $360° = \mp \infty$, *and all the real values of this function are comprised between* 1 *and* $\pm \infty$, *and* -1 *and* $\mp \infty$.

Dem.—Let the student demonstrate and illustrate as in the preceding article. Do not neglect to go through the whole in detail; it is an important and excellent exercise.

39. Prop.—*Versin* $0° = 0$, *versin* $90° = 1$, *versin* $180° = 2$, *versin* $270° = 1$, *versin* $360° = 0$, *and the real limits of the function are* 0 *and* 2.

Dem.—The student will readily deduce these results from the relation $\text{vers}\, x = 1 - \cos x$. Thus when $x = 0$, $\text{vers}\, 0 = 1 - \cos 0 = 1 - 1 = 0$, etc.

---

40. Prop.—*Covers 0° = 1, covers 90° = 0, covers 180° = 1, covers 270° = 2, covers 360° = 1, and the limits of the real values of this function are 0 and 2.*

Dem.—The student should be able to give it.

---

41. General Scholium.—It is important to observe that in the case of each of the above functions *it changes its sign by passing through 0 or ∞.* In fact, it is assumed, in mathematics, that a varying quantity which passes from + to −, does so by passing through 0 or ∞. The converse, however, is by no means true; viz., that whenever a varying quantity passes through 0 or ∞, its sign necessarily changes.*

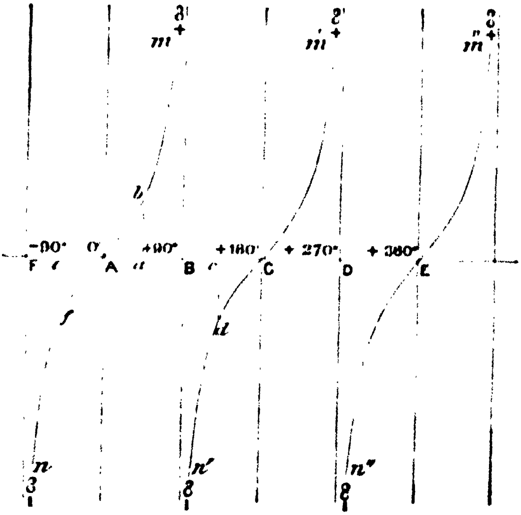


Fig. 8.

* The Co-ordinate Geometry affords elegant illustrations of the theory of the change in value and sign of these functions. (See Gen. Geom., **23**, etc.) The annexed Figure represents a curve, (or as the student may be disposed to consider it, a *series of curves*), constructed as follows: On the indefinite line **AE**, a circumference is developed (as it were straightened out), the origin being at **A**, and **AE** being the length of the circumference. The curves *mn*, *m'n'*, *m''n''* are drawn by erecting at every few degrees from **A**, a line equal to the tangent of the same number of degrees, *above* the line **AE** when the tangent is +, and *below* **AE** when the tangent is −. Thus **A***a* = 45° in length, and *ab* = tan 45°; **A***c* = 135°, and *cd* = tan 135°. Such lines as *ab*, *cd*, etc., are called ordinates of the curve. The law of change in these ordinates is manifestly the same as the law of change in tangents. We see that as we pass from 0° to 90°, the ordinate (tangent) passes from 0 to + ∞. At 90° the ordinate (tangent) is *both* + and −, *i. e.*, ± ∞. So also at 270°, and at other similar points. A similar device illustrates the changes in the other trigonometrical functions. Some may see the propriety of distinguishing ∞ as + and −, who, nevertheless, do not see why it is necessary to make the same distinction in the case of 0. But a moment's reflection will show that one distinction involves the other, since ∞ and 0 are mutually reciprocals of each other.

42. SCH. The results of the preceding discussion of the signs and limits of the trigonometrical functions, **24** to **41**, are exhibited in the annexed

TABLE.

x	SIN x	COS x	TAN x	COT x	SEC x	COSEC x	VERSIN x	COVER-SIN x	
$0°$ to $\frac{\pi}{2}$	∓ 0 to $+1$	$+1$ to ± 0	∓ 0 to $\pm \infty$	$\mp \infty$ to ± 0	$+1$ to $\pm \infty$	$\mp \infty$ to $+1$	$+0$ to $+1$	$+1$ to $+0$	*At* the Limits.
	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	*Between* "
$\frac{\pi}{2}$ to π	$+1$ to ± 0	± 0 to -1	$\pm \infty$ to ∓ 0	± 0 to $\mp \infty$	$\pm \infty$ to -1	$+1$ to $\pm \infty$	$+1$ to $+2$	$+0$ to $+1$	*At* the Limits.
	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	*Between* "
π to $\frac{3\pi}{2}$	± 0 to -1	-1 to ∓ 0	∓ 0 to $\pm \infty$	$\mp \infty$ to ± 0	-1 to $\mp \infty$	$\pm \infty$ to -1	$+2$ to $+1$	$+1$ to $+2$	*At* the Limits.
	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$+$	*Between* "
$\frac{3\pi}{2}$ to 2π	-1 to ∓ 0	∓ 0 to $+1$	$\pm \infty$ to ∓ 0	± 0 to $\mp \infty$	$\mp \infty$ to $+1$	-1 to $\mp \infty$	$+1$ to $+0$	$+2$ to $+1$	*At* the Limits.
	$-$	$+$	$-$	$-$	$+$	$-$	$+$	$+$	*Between* "

SEE ART. 32, SCH'S 1 and 2.

TRIGONOMETRICAL FUNCTIONS OF NEGATIVE ARCS (OR ANGLES).

43. Prop.—*Changing the sign of an arc (or angle) changes the sign of its sine, and consequently of its cosecant; i. e.,* $\sin(-x) = -\sin x$,[*] and $\operatorname{cosec}(-x) = -\operatorname{cosec} x$.

DEM.—1st, If the angle (or arc) is numerically *less* than 90°, and +, it ends in the first quadrant, and hence its sine is +; but, if the angle (or arc) is numerically *less* than 90° and −, it ends in the fourth quadrant, and hence its sine is −. That is, x being numerically less than 90°, $\sin(-x) = -\sin x$. 2d, If x is between 90° and 180° in numerical value, the arc ends in the second quadrant when x is +, and hence its sine is +; but when x is −, the arc ends in the third quadrant, whence its sine is −. ∴ In this case, also, $\sin(-x) = -\sin x$. 3d, If x is between 180° and 270° in numerical value and +, the arc ends in the third quadrant, whence $\sin x$ is −, $(-\sin x)$; but if the arc is −, it ends in the second quadrant, whence its sign is +, $[+\sin(-x)]$. ∴ In this case $-\sin x = \sin(-x)$, or $\sin x = -\sin(-x)$. 4th, In like manner the student will observe that $-\sin x = \sin(-x)$, or $\sin x = -\sin(-x)$, when x is between 270° and 360°. Moreover, since this order will recur as we pass around again, we learn that in any case the sign of the sine of a negative angle is the opposite of that of an equal positive angle. Finally, $\operatorname{cosec}(-x) = \dfrac{1}{\sin(-x)} = \dfrac{1}{-\sin x} = -\dfrac{1}{\sin x} = -\operatorname{cosec} x$. Q. E. D.

44. Prop.—*Changing the sign of an angle (or arc) does not change the sign of its cosine or secant; i. e.,* $\cos(-x) = \cos x$, and $\sec(-x) = \sec x$.

DEM.—1st, When $x < 90°$ and +, the arc ends in the first quadrant, and hence its cosine is +; so, also, if $x < 90°$ and −, though the arc ends in the fourth quadrant, its cosine is still +. ∴ $\cos(-x) = \cos x$. (The student can supply the other three cases.) Finally, $\sec(-x) = \dfrac{1}{\cos(-x)} = \dfrac{1}{\cos x} = \sec x$. Q. E. D.

45. Prop.—*Changing the sign of an angle (or arc) changes the sign of its tangent, and consequently of its cotangent; i. e.,* $\tan(-x) = -\tan x$, and $\cot(-x) = -\cot x$.

---

[*] The student should be careful to note the exact meaning of this expression. It is read "sine minus x = minus sine x," and means that the sine of a negative angle (or arc) is equal (numerically) to the sine of the same positive angle (or arc), but has the opposite sign.

Dem.—This is an immediate consequence of the fact that changing the sign of the angle (or arc) changes the sign of its sine, but *not* of its cosine. Thus, $\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\frac{\sin x}{\cos x} = -\tan x$. Also, $\cot(-x) = \frac{1}{\tan(-x)}$ $= \frac{1}{-\tan x} = -\frac{1}{\tan x} = -\cot x$; or $\cot(-x) = \frac{\cos(-x)}{\sin(-x)} = \frac{\cos x}{-\sin x} = -\frac{\cos x}{\sin x}$ $= -\cot x$.

Sch.—The proper sign of a function of a negative angle can always be ascertained by observing in what quadrant the measuring arc ends, in a manner altogether similar to that in which the signs of the functions of positive angles are determined.

CIRCULAR FUNCTIONS.

46. Circular Functions are angles (or arcs) expressed as functions of sines, cosines, tangents, or other trigonometrical lines.

Ill's.—In the expression sin x, we designate a *sine*, *i. e.*, a right line, merely using the x to tell *what* sine, as the sine of 20°, of 135°, etc. But we often wish to speak of an angle (or arc) which has a particular sine, tangent, or other trigonometrical line. Thus, we say, "the angle (or arc) whose sine is $\frac{1}{2}$," "the angle (or arc) whose tangent is 3," etc. In this form of expression, it is evidently the angle (or arc) which is the thing mainly thought of; and it is conceived as dependent upon its trigonometrical line.

47. Notation.—The circular functions are written $\sin^{-1}y$, $\cos^{-1}x$, $\tan^{-1}z$, etc.; and are read "the angle (or arc) whose sine is y," "the angle whose tangent is z," etc.

Ill's.—The expressions $x = \sin^{-1}y$, and $y = \sin x$, are ultimately equivalent, since the first is "x = angle whose sine is y," and the second, "y = the sine of x." The only difference is, that in the first form the *angle* (x) is the thing thought of, and the sine (y) is used merely to tell *what* angle; but in the second form, the *sine* (y) is the prominent thing, and the *angle* (x) is used simply to tell *what* sine. This mutual relation has caused the circular functions to be called also *Inverse Functions.*

Sch.—This notation is rather an unfortunate one, inasmuch as it is the same as has been already adopted in the theory of exponents. The student will however observe that the signification in this instance is altogether different from the former. Thus, since we write "the square of sine x," $\sin^2 x$; according to the theory of exponents, $\sin^{-2}x$ would be $\frac{1}{\sin^2 x}$. So also $\sin^{-1}x$ should mean $\frac{1}{\sin x}$. Now, the former of these expressions would actually signify as indicated (though it were better to write it $(\sin x)^{-2}$), while the latter does not mean at all what the theory of exponents would make it. Unfortunate as the notation is, it is probably best to retain it. It, doubtless, was suggested thus: If we have $y = a^2x$, we may write $x = a^{-2}y$, so also $y = a x$, may be written $x = a^{-1}y$. This affords a parallelism in form, but not in

EXERCISES.

1. What is the complement of 150° 21′ 13″.5? What the supplement? Give the complements and also the supplements of 125° 15′, 283° 21′ 11″, 36° 05′ 02″, and 89° 00′ 12″.

2. How many degrees in the angle (or arc) $\frac{\pi}{3}$? In ¾π? In the arc 2π? In ¼π? In 1 ¼π?

3. How many times is the radius contained in 108°? How many times in 2π? How many in 460°? How many in 210°?

4. Radius being taken as the measure of the arc, by what are 45° represented? By what 90°? By what 180°? By what 225°? How many degrees does 1 represent, radius being the measure?

5. Find the length of a degree of the meridian upon a globe of 18 inches diameter.

6. Express 12° 22′ 13″ 11‴ 05⁗ in °, ′, ″, and decimals of a second. So also express 53°.51 in °, ′, ″, etc.

7. How many degrees, minutes, and seconds in an arc equal to twice radius? three times radius? Show that 27° is equivalent to $\frac{3}{20}\pi$. That 10° to radius 10 ft. = 1.75 ft. What radius gives 1″ = 1 inch?

8. Draw any angle, and construct its sine, cosine, tangent, or any other trigonometrical function, and then determine as nearly as practicable the numerical values of the function by actual measurement.

Solution.—Given the angle **MON**, to find the numerical value of the tangent as near as practicable by measurement. Taking any convenient unit, as **OA**, for a radius, and striking the arc **A***a*, draw **AT** tangent to *a***A** at **A**. Now apply **OA** to **AT** and find their ratio (Part I., *36*). In this case **AT** = 1½ approximately, ∴ tan **MON** = 1½.

[Note.—The student should practice upon such examples, finding the values of all the functions until the process, and the meaning of the numerical value of any function of an angle, are clearly seen.]

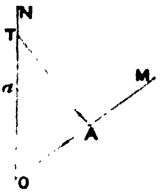


Fig. 9.

9. Construct an angle whose sine is ⅔, *i. e.*, $\sin^{-1}\frac{2}{3}$.

Solution.—Let **O** be the required angular point, and **OA** one side of the angle. Lay off from **O** on **OA** 3 measures of any convenient length, making **O***a*. Using **O***a* as a radius, describe the indefinite arc *a***M**. Erect **OC** perpendicular to **OA** and take **OC** = ⅔ of **O***a*. Through **C** draw **CP** parallel to **OA**. Finally, draw **OB** through **P**. **AOB** is the angle required, since **O***a* being 1, the sine of **AOB**, **PD**, is ⅔. **AOB** = $\sin^{-1}\frac{2}{3}$.

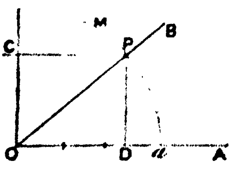


Fig. 10.

10. Construct an angle whose cosine is $\frac{2}{3}$. That is, construct $\cos^{-1}\frac{2}{3}$.

SUG.—The construction is the same as in the last example, except that instead of OC being drawn to limit the arc, a perpendicular is erected to Oa at $\frac{2}{3}$ (the second point of division) from O, and the point P located where this perpendicular intersects the arc.

11. Construct an angle whose tangent is 2. That is, construct $\tan^{-1}2$.

SUG.—To construct this at O on the line OA, *Fig.* 10, take any convenient radius, Oa, and strike the indefinite arc. Then erect at a a tangent, and make it equal to twice the radius used. Through the extremity of this tangent and O draw a line, and the angle between this and OA is $\tan^{-1}2$.

12. Construct $\sec^{-1}2$; $\cot^{-1}3$; $\operatorname{cosec}^{-1}1\frac{1}{2}$; an obtuse angle $\sin^{-1}\frac{1}{2}$; $\tan^{-1}(-3)$.

SUG.—To construct $\sin^{-1}\frac{1}{2}$, see *Fig.* 7. Let OA be one side of the angle, and O the vertex. With any convenient radius draw the semicircumference AaB, and draw the perpendicular Oa. Bisect this perpendicular, and through the point of bisection draw a parallel to AB, intersecting the arc in the 2d quadrant. Through this intersection draw a line, as OP′. Then AOP′ (assuming the construction as specified, and not as in the figure) $= \sin^{-1}\frac{1}{2}$.

13. Construct the following: $\tan^{-1}1$; $\tan^{-1}(-1)$; $\tan^{-1}\frac{3}{4}$; $\tan^{-1}(-2)$; $\tan^{-1}(-\frac{1}{2})$; $\cos^{-1}(-\frac{1}{3})$; $\sec^{-1}(-2)$; $\operatorname{cosec}^{-1}(-3)$; $\operatorname{versin}^{-1}\frac{1}{2}$; $\operatorname{versin}^{-1}1\frac{1}{3}$.

14. From the fundamental relations (*21*) deduce the following: $\sin x = \sqrt{1-\cos^2 x}$; $\cos x = \sqrt{1-\sin^2 x}$; $\tan x \cot x = 1$; $\tan x \cos x = \sin x$; $\cos x = \frac{\sin x}{\tan x}$; $\sin x = \frac{\cos x}{\cot x}$; $\sin x = \frac{1}{\sqrt{1+\cot^2 x}}$; $\cos x = \frac{1}{\sqrt{1+\tan^2 x}}$; $\tan x = \sin x \sec x$; $\cot x = \cos x \operatorname{cosec} x$; $\tan x = \frac{\sec x}{\operatorname{cosec} x}$; $\sin x = \frac{\cos x}{\sqrt{\operatorname{cosec}^2 x - 1}}$; $\operatorname{vers} x = \frac{\sec x - 1}{\sec x} = \frac{\operatorname{cosec} x - \cot x}{\operatorname{cosec} x}$; $\cot x = \frac{\sqrt{1-\sin^2 x}}{\sin x}$.

15. Given $\tan x = \frac{3}{4}$, to find the other trigonometrical functions of x.

Results: Sec $x = \frac{5}{4}$; $\cos x = \frac{4}{5}$; $\sin x = \frac{3}{5}$; $\operatorname{cosec} x = \frac{5}{3}$; $\cot x = \frac{4}{3}$; $\operatorname{vers} x = \frac{1}{5}$; $\operatorname{covers} x = \frac{2}{5}$.

16. Given $\sin x = \frac{3}{5}$, to find the other trigonometrical functions of x.

17. Given $\sec x = 2$, to find the other trigonometrical functions of x.

18. Given $\tan x = -1$, to find the other trigonometrical functions of x.

Results.—Cot $x = -1$; $\sec x = \mp\sqrt{2}$; cosec $x = \pm\sqrt{2}$; $\sin x = \pm \frac{1}{2}\sqrt{2}$; $\cos x = \mp\frac{1}{2}\sqrt{2}$; versin $x = 1 \pm \frac{1}{2}\sqrt{2}$; coversin $x = 1 \mp \frac{1}{2}\sqrt{2}$.

[NOTE.—Observe closely the signs of the functions in Ex. 18.]

19. In the preceding examples the constructions required have been limited to angles less than 180°, but it is evident that an infinite number of angles (or arcs) according to the more comprehensive trigonometrical view, correspond to the same function. What angles or arcs have their tangents each 1? What, each -1? Construct an angle between 180° and 270°, whose sine is $-\frac{1}{2}$. What other angle less than 360° has the same sine?

20. Having given a sine, how many angles less than 180° correspond to it? Construct the angle or angles less than 180° whose sine is $\frac{3}{4}$. How many angles less than 180° have the same cosine? tangent? cotangent? (In each of the last three cases *only one*.) Construct $y = \cos^{-1}\frac{1}{2}$; $y = \cos^{-1}(-\frac{1}{2})$. How are these angles related to each other? Construct $y = \tan^{-1}3$; $y = \tan^{-1}(-3)$. How are these angles related to each other? (Restrict the constructions and questions in this example to angles less than 180°.)

21. Given $y = \sin^{-1}x$, show that $\cos y = \sqrt{1-x^2}$; $\tan y = \dfrac{x}{\sqrt{1-x^2}}$; $\sec y = \dfrac{1}{\sqrt{1-x^2}}$; $y = \operatorname{cosec}^{-1}\dfrac{1}{x}$; vers $y = 1 - \sqrt{1-x^2}$; $y = \cot^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$.

22. What are trigonometrical functions of 450°? Of 1350°? Of 900°?

23. Show that the following are true for all integral values of n, including 0: $\sin 4n\frac{\pi}{2} = 0$; $\sin(4n+1)\frac{\pi}{2} = 1$; $\sin(4n+2)\frac{\pi}{2} = 0$; $\sin(4n+3)\frac{\pi}{2} = -1$; $\cos 4n\frac{\pi}{2} = 1$; $\cos(4n+1)\frac{\pi}{2} = 0$; $\cos(4n+2)\frac{\pi}{2} = -1$; $\cos(4n+3)\frac{\pi}{2} = 0$; $\tan 2n\frac{\pi}{2} = 0$; $\tan(2n+1)\frac{\pi}{2} = \infty$.

24. What are the signs of the several trigonometrical functions of $-110°$? Of $-35°$? Of $-500°$? Of -2000?

25. Prove that $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{1}{2}\sqrt{3}$, $\tan 30° = \frac{1}{3}\sqrt{3}$, and $\cot 30° = \sqrt{3}$, $\sec 30° = \frac{2}{3}\sqrt{3}$, and cosec $30° = 2$.

SUG.—Observe that the chord of 60° = 1, and that the sine of 30° = ½ the chord of 60°. Make the figure.

26. Prove that $\sin 45° = \frac{1}{2}\sqrt{2}$, $\cos 45° = \frac{1}{2}\sqrt{2}$, $\tan 45° = 1$, $\cot 45° = 1$, $\sec 45° = \sqrt{2}$, and $\operatorname{cosec} 45° = \sqrt{2}$.

SUG.—Observe that $\sin 45° = \cos 45°$; hence $\sin^2 45° + \cos^2 45° = 1$, becomes $2\sin^2 45° = 1$.

SCH.—The values obtained in the last two examples should be retained in the memory, as they are of frequent use. The functions of 30° and of 45° are always assumed to be known in any trigonometrical operation.

SECTION II.

RELATIONS BETWEEN THE TRIGONOMETRICAL FUNCTIONS OF DIFFERENT ANGLES (OR ARCS).

(*a*) FUNCTIONS OF THE SUM OR DIFFERENCE OF ANGLES (OR ARCS).

48. Prop.—*The sine of the sum of two angles (or arcs) is equal to the sine of the first into the cosine of the second, plus the cosine of the first into the sine of the second.* Thus letting x and y represent any two angles (or arcs),

$$\sin(x + y) = \sin x \cos y + \cos x \sin y.$$

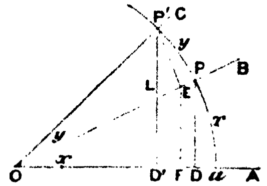


FIG. 11.

DEM.—Let AOB and BOC be the two angles represented respectively by x and y. Draw the measuring arc aP′, and the sines PD, and P′E of the angles. AOC = AOB + BOC, is the *sum* of the two angles. Draw P′D′, *the sine of the sum* of the two angles. Then PD = $\sin x$, P′E = $\sin y$, OD = $\cos x$, OE = $\cos y$, P′D′ = $\sin(x + y)$, and OD′ = $\cos(x + y)$.

Now $\sin(x + y)$ = P′D′ = EF + P′L. But from the similar triangles EOF, and POD, we have $\frac{EF}{OE} = \frac{PD}{OP}$, or $\frac{EF}{\cos y} = \frac{\sin x}{1}$. ∴ EF = $\sin x \cos y$. Also, from the similar triangles P′EL and POD, we have $\frac{P'L}{P'E} = \frac{OD}{OP}$, or $\frac{P'L}{\sin y} = \frac{\cos x}{1}$. ∴ P′L = $\cos x \sin y$. Substituting these values of EF and P′L, we have $\sin(x + y) = \sin x \cos y + \cos x \sin y$. Q. E. D.*

---

* This demonstration may seem defective, since the sum of the angles x and y, as represented in the diagram, is less than 90°; nevertheless, in the General (Analytical) Geometry, we constantly proceed in a manner entirely analogous; viz., first produce the equation of a locus from some particular figure, and then make it general in application. The demonstrations in cases in which the sum of the angles is greater than 90°, etc., are similar, and some of them will be given in the EXERCISES at the close of the section. It is not thought best to cumber the purely theoretical part of the subject with such matter.

Cor. $Sin(90^\circ + x) = \cos x$. $Sin(180^\circ + x) = -\sin x$. $Sin(270^\circ + x) = -\cos x$. $Sin(360^\circ + x) = \sin x$.

Dem.—From the proposition we have,

$$\sin(90^\circ + x) = \sin 90^\circ \cos x + \cos 90^\circ \sin x = \cos x;$$

since $\sin 90^\circ = 1$, and $\cos 90^\circ = 0$.

In like manner,

$$\sin(180^\circ + x) = \sin 180^\circ \cos x + \cos 180^\circ \sin x = -\sin x;$$

since $\sin 180^\circ = 0$, and $\cos 180^\circ = -1$.

[The student will readily produce the other forms.]

49. Prop.—*The sine of the difference between two angles (or is equal to the sine of the first into the cosine of the second, minus the cosine of the first into the sine of the second.* Thus, letting x and y represent the angles,

$$\sin(x - y) = \sin x \cos y - \cos x \sin y.$$

Dem.—In $\sin(x + y) = \sin x \cos y + \cos x \sin y$, substituting $-y$ for y, we have, $\sin(x - y) = \sin x \cos(-y) + \cos x \sin(-y) = \sin x \cos y - \cos x \sin y$; since $\cos(-y) = \cos y$, and $\sin(-y) = -\sin y$. (**43, 44.**)

Cor. $Sin(90^\circ - x) = \cos x$. $Sin(180^\circ - x) = \sin x$. $Sin(270^\circ - x) = -\cos x$. $Sin(360^\circ - x) = -\sin x$.

Dem.—This is simply an application of the proposition, as the preceding corollary is of its proposition. (The student will make it.) Or we may deduce the results from the corollary under the preceding proposition by merely substituting $-x$ for x. Thus, $\sin(90^\circ - x) = \cos(-x) = \cos x$; $\sin(180^\circ - x) = -\sin(-x) = -(-\sin x) = \sin x$; etc.

50. Prop.—*The cosine of the sum of two angles (or arcs) is equal to the rectangles of their cosines, minus the rectangle of their sines.* Thus, letting x and y represent the angles,

$$\cos(x + y) = \cos x \cos y - \sin x \sin y.$$

Dem.—Taking the formula $\sin(x - y) = \sin x \cos y - \cos x \sin y$, and substituting $90^\circ - x$ for x, we have, $\sin(90^\circ - x - y) = \sin(90^\circ - x)\cos y - \cos(90^\circ - x)\sin y$. Now $90^\circ - x - y = 90^\circ - (x + y)$; and $\sin[90^\circ - (x + y)] =$ the sine of the complement of $(x + y)$, or $\cos(x + y)$. Also $\sin(90^\circ - x) = \cos x$ and $\cos(90^\circ - x) = \sin x$. $\therefore$ Substituting, $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

Cor. $Cos(90^\circ + x) = -\sin x$. $Cos(180^\circ + x) = -\cos x$. $Cos(270^\circ + x) = \sin x$. $Cos(360^\circ + x) = co$

Dem.—Apply the proposition.

51. Prop.—*The cosine of the difference of two angles (or arcs) is equal to the rectangle of their cosines, plus the rectangle of their sines.* Thus, letting x and y represent the angles,

$$\cos(x-y) = \cos x \cos y + \sin x \sin y.$$

Dem.—In $\cos(x+y) = \cos x \cos y - \sin x \sin y$, substituting $-y$ for y, we have, $\cos(x-y) = \cos x \cos(-y) - \sin x \sin(-y)$, or $\cos(x-y) = \cos x \cos y + \sin x \sin y$; since $\cos(-y) = \cos y$, and $\sin(-y) = -\sin y$. [Notice that the last term becomes $-\sin x(-\sin y)$, which equals $+\sin x \sin y$.]

Cor. *$Cos(90°-x) = \sin x$. $Cos(180°-x) = -\cos x$. $Cos(270°-x) = -\sin x$. $Cos(360°-x) = \cos x$.*

Dem.—Apply the proposition.

---

52. Prop.—*The tangent of the sum of two angles (or arcs) is equal to the sum of their tangents, divided by 1 minus the rectangle of their tangents.* Thus, x and y being the angles, we have,

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

Dem. $\text{Tan}(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$. Dividing numerator and denominator of the last fraction by $\cos x \cos y$,* we have, $\tan(x+y) = \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 - \frac{\sin x \sin y}{\cos x \cos y}} = \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 - \frac{\sin x}{\cos x} \times \frac{\sin y}{\cos y}} = \frac{\tan x + \tan y}{1 - \tan x \tan y}$; since $\frac{\sin x}{\cos x} = \tan x$, etc., and $\frac{\sin x \sin y}{\cos x \cos y} = \frac{\sin x}{\cos x} \times \frac{\sin y}{\cos y}$. Q. E. D.

Cor. *$Tan(90°+x) = -\cot x$. $Tan(180°+x) = \tan x$. $Tan(270°+x) = -\cot x$. $Tan(360°+x) = \tan x$.*

---

* The three following forms may readily be obtained by dividing respectively by $\sin x \sin y$, $\sin x \cos y$, and $\cos x \sin y$: viz., $\tan(x+y) = \frac{\cot y + \cot x}{\cot x \cot y - 1}$, $\tan(x+y) = \frac{1 + \cot x \tan y}{\cot x - \tan y}$, and $\tan(x+y) = \frac{\tan x \cot y + 1}{\cot y - \tan x}$. Any one of these may be reduced to the one given in the proposition, by substituting in it $\frac{1}{tan}$ for *cot*, and reducing. Notice that the form in the *Prop.* is in terms of the tangents. Also observe *why* dividing by a certain term gives a particular form, and by which to divide to get a required form.

Dem. $\text{Tan}\,(90^\circ + x) = \dfrac{\sin(90^\circ + x)^*}{\cos(90^\circ + x)} = \dfrac{\cos x}{-\sin x} = -\cot x$. $\text{Tan}\,(180^\circ + x) = \dfrac{\sin(180^\circ + x)}{\cos(180^\circ + x)} = \dfrac{-\sin x}{-\cos x} = \tan x$, etc., etc.

53. *Prop.*—*The tangent of the difference of two angles (or arcs) is equal to the difference of their tangents, divided by* 1 *plus the rectangle of their tangents.* Thus, x and y being the angles,

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.$$

Demonstration. $\text{Tan}\,(x - y) = \dfrac{\sin(x - y)}{\cos(x - y)} = \dfrac{\sin x \cos y - \cos x \sin y}{\cos x \cos y + \sin x \sin y} = \dfrac{\dfrac{\sin x \cos y}{\cos x \cos y} - \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} + \dfrac{\sin x \sin y}{\cos x \cos y}} = \dfrac{\dfrac{\sin x}{\cos x} - \dfrac{\sin y}{\cos y}}{1 + \dfrac{\sin x}{\cos x} \times \dfrac{\sin y}{\cos y}} = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$. Q. E. D. (See foot-notes to preceding proposition.) This proposition is also readily deduced from the preceding by substituting in the formula $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$, $-y$ for y, and remembering that $\tan(-y) = -\tan y$.

Cor. *Tan*$(90^\circ - x) = $ *cot* x. *Tan*$(180^\circ - x) = -$*tan* x. *Tan*$(270^\circ - x) = $ *cot* x. *Tan* $(360^\circ - x) = -$ *tan* x.

Dem. $\text{Tan}\,(90^\circ - x) = \dfrac{\sin(90^\circ - x)}{\cos(90^\circ - x)} = \dfrac{\cos x}{\sin x} = \cot x$. $\text{Tan}\,(180^\circ - x) = \dfrac{\sin(180^\circ - x)}{\cos(180^\circ - x)} = \dfrac{\sin x}{-\cos x} = -\tan x$. $\text{Tan}\,(270^\circ - x) = \dfrac{\sin(270^\circ - x)}{\cos(270^\circ - x)} = \dfrac{-\cos x}{-\sin x} = \cot x$. $\text{Tan}\,(360^\circ - x) = \dfrac{\sin(360^\circ - x)}{\cos(360^\circ - x)} = \dfrac{-\sin x}{\cos x} = -\tan x$.

---

[*] These and the kindred *formulæ* may be produced by a direct application of the proposition. Thus, $\tan(90^\circ + x) = \dfrac{\tan 90^\circ + \tan x}{1 - \tan 90^\circ \tan x} = \dfrac{\tan 90^\circ}{-\tan 90^\circ \tan x} = \dfrac{1}{-\tan x} = -\cot x$. (The reason for dropping $\tan x$ and 1 is that they are finite terms connected with infinities, as $\tan 90^\circ = \infty$. Or, otherwise, $\tan(90^\circ + x) = \dfrac{\tan 90^\circ + \tan x}{1 - \tan 90^\circ \tan x} = \dfrac{1 + \dfrac{\tan x}{\tan 90^\circ}}{\dfrac{1}{\tan 90^\circ} - \tan x} = \dfrac{1 + \dfrac{\tan x}{\infty}}{\dfrac{1}{\infty} - \tan x} = \dfrac{1}{-\tan x}$, since a finite divided by an infinite equals 0. Again, $\tan(180^\circ + x) = \dfrac{\tan 180^\circ + \tan x}{1 - \tan 180^\circ \tan x} = \dfrac{\tan x}{1} = \tan x$, since $\tan 180^\circ = 0$, etc., etc.

54. Prop.—*The cotangent of the sum of two angles (or arcs) is equal to the rectangle of their cotangents minus* 1, *divided by their sum.* Thus, x and y being the angles,

$$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}.$$

Dem. $$\mathrm{Cot}(x+y) = \frac{\cos(x+y)}{\sin(x+y)} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y} = \frac{\dfrac{\cos x \cos y}{\sin x \sin y} - \dfrac{\sin x \sin y}{\sin x \sin y}}{\dfrac{\sin x \cos y}{\sin x \sin y} + \dfrac{\cos x \sin y}{\sin x \sin y}}$$

$$\therefore \frac{\dfrac{\cos x}{\sin x} \times \dfrac{\cos y}{\sin y} - 1}{\dfrac{\cos y}{\sin y} + \dfrac{\cos x}{\sin x}} = \frac{\cot x \cot y - 1}{\cot y + \cot x}.$$ Q. E. D. Or we may deduce it thus,

$$\cot(x+y) = \frac{1}{\tan(x+y)} = \frac{1 - \tan x \tan y}{\tan x + \tan y} = \frac{1 - \dfrac{1}{\cot x \cot y}}{\dfrac{1}{\cot x} + \dfrac{1}{\cot y}} = \frac{\cot x \cot y - 1}{\cot y + \cot x}.$$

Q. E. D.

Cor. $Cot(90° + x) = - tan\ x$. $Cot(180° + x) = cot\ x$. $Cot(270° + x) = - tan\ x$. $Cot(360° + x) = cot\ x$.

Dem. Divide cosine by sine, or take the reciprocals of the corresponding tangents. Thus, $\cot(90° + x) = \dfrac{1}{\tan(90° + x)} = \dfrac{1}{-\cot x} = -\tan x$, etc.

---

55. Prop.—*The cotangent of the difference of two angles (or arcs) is equal to the rectangle of their cotangents plus* 1, *divided by their difference.* Thus, x and y being the angles,

$$\cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}.$$

Dem. Substitute $-y$ for y, in the preceding formula; or, divide $\cos(x-y)$ by $\sin(x-y)$ and reduce; or, take the reciprocal of $\tan(x-y)$, and substitute $\dfrac{1}{\cot}$ for *tan*.

Cor. $Cot(90° - x) = tan\ x$. $Cot(180° - x) = - cot\ x$. $Cot(270° - x) = tan\ x$. $Cot(360° - x) = - cot\ x$.

Dem. Same as above.

Sch. 1. The *formulæ* for the secant and cosecant of the sum and of the difference of two angles (or arcs) are not of sufficient importance to warrant their introduction here; some of them will be given in the exercises, as also the extension of those already given to the case of the sum of three or more angles, or arcs.

Sch. 2. The results reached in this discussion are so important that we will collect them into a

TABLE.

(A) $\sin(x + y) = \sin x \cos y + \cos x \sin y.$
(B) $\sin(x - y) = \sin x \cos y - \cos x \sin y.$
(C) $\cos(x + y) = \cos x \cos y - \sin x \sin y.$
(D) $\cos(x - y) = \cos x \cos y + \sin x \sin y.$

$$\text{(E)}\quad \tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

$$\text{(F)}\quad \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.$$

$$\text{(G)}\quad \cot(x + y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}.$$

$$\text{(H)}\quad \cot(x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}.$$

(I)

x	$90° - x$	$90° + x$	$180° - x$	$180° + x$	$270° - x$	$270° + x$	$360° - x$	$360° + x$
sine	$\cos x$	$\cos x$	$\sin x$	$-\sin x$	$-\cos x$	$-\cos x$	$-\sin x$	$\sin x$
cosine	$\sin x$	$-\sin x$	$-\cos x$	$-\cos x$	$-\sin x$	$\sin x$	$\cos x$	$\cos x$
tangent	$\cot x$	$-\cot x$	$-\tan x$	$\tan x$	$\cot x$	$-\cot x$	$-\tan x$	$\tan x$
cotangent	$\tan x$	$-\tan x$	$-\cot x$	$\cot x$	$\tan x$	$-\tan x$	$-\cot x$	$\cot x$

It will not be found difficult to memorize and extend set (I), if the student observes, that, when the number of whole quadrants is *odd* (as 90°, 270°, etc.), the function *changes name* (as from sin to cos, from cos to sin, etc.); but, when the number of whole quadrants is *even*, the function retains the *same* name. The sign of the sine and cosine is readily determined according to fundamental principles by observing where the arc ends, assuming $x < 90°$. Thus $180° + x$ ends in the third quadrant; hence its sine (which in numerical value is $\sin x$) is $-$, and its cosine is also $-$. As the signs of the tangent and cotangent of the same arc are alike, we have only to observe whether the sine and cosine, in any given case, have like or unlike signs, in order to determine the sign of the tangent and cotangent. For example, what is $\cot(630° + x)$ equal to? The number of quadrants being odd (7), the function changes *name*, and since the arc ends in the fourth quadrant, its sine is $-$, and its cosine $+$; therefore $\cot(630° + x) = -\tan x$. If in any given case $x > 90°$, determine the character of the function as above, on the hypothesis $x < 90°$, and then modify the result for the particular value of x. Thus in the last case, if x was between 90° and 180°, its tangent would be $-$, and for such a value $\cot(630° + x) = \tan x$. Or, we may consider at first where the arc ends, taking into consideration the given value of x.

(*b*) FUNCTIONS OF DOUBLE AND HALF ANGLES.

56. Prop.—*Letting x represent any angle (or arc),*

(K) $\sin 2x = 2\sin x \cos x$; (M) $\tan 2x = \frac{}{1 -}$

(L) $\cos 2x = \cos^2 x - \sin^2 x =$
$2\cos^2 x - 1$, or $1 - 2\sin^2 x$;

DEM. These results are readily deduced from (A), (C), (E), and (G) Thus, in $\sin(x + y) = \sin x \cos y + \cos x \sin y$, if we make $y = x$, we have $\sin 2x = \sin x \cos x + \cos x \sin x = 2\sin x \cos x$. (In like manner produce the others.)

7. Prop.—*Letting x represent any angle (or arc), we have,*

(O) $\sin \frac{1}{2}x = \pm\sqrt{\frac{1}{2}(1 - \cos x)}$; (Q)

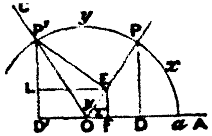
$1 - \cos x$

DEM. From (L), $2\sin^2 x = 1 - \cos 2x$, or $\sin x = \pm\sqrt{\frac{1}{2}(1 - \cos 2x)}$. Putting $\frac{1}{2}x$ for x, this becomes $\sin \frac{1}{2}x = \pm\sqrt{\frac{1}{2}(1 - \cos x)}$. In like manner, from the same formula (L), $2\cos^2 x = 1 + \cos 2x$; whence, $\cos \frac{1}{2}x = \pm\sqrt{\frac{1}{2}(1 + \cos x)}$. Again, $\tan \frac{1}{2}x = \frac{\sin \frac{1}{2}x}{\cos \frac{1}{2}x} \quad \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}}$; and $\cot \frac{1}{2}x = \frac{1}{\tan \frac{1}{2}x} = \pm\sqrt{\frac{1 + \cos x}{1 - \cos x}}$. Q. E. D.

SCH. The sign of the function in the case of each of these is + if $x < 180°$; but can only be determined by the value of x in any given case.

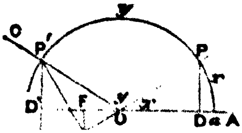
EXERCISES.

1. Prove from *Fig.* (*a*) that $\sin(x + y) = \sin x \cos y + \cos x \sin y$, when x and y are each $< 90°$, but $x + y > 90°$.



2. Same as in *Ex.* 1, from (*b*), when $x < 90°$, $x + y > 90°$, and $< 180°$, and $y > 90°$ and $< 180°$.

SUG. In this case, $\sin(x + y) = P'D' = P'L - EF$. In other respects the demonstration is identical with the preceding. This gives $\sin(x + y) = \cos x \sin y - \sin x \cos y$. But the

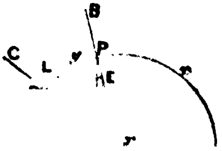


— sign is accounted for in the general formula, $\sin(x + y) = \sin x \cos y + \cos x \sin y$, by noticing that $\cos y$ is —, when $y > 90°$ and $< 180°$.

3. Same as in the preceding, when $x > 90°$, $y < 90°$, and $(x + y) < 180°$.

Sug. Here $\sin(x + y) =$ P′D′ = EF — P′L. In all other respects the demonstration is identical with the other cases. The — sign in this case arises from x being between 90° and 180°, whence $\cos x$ is —.

[Note. A number of other cases may be devised, but the above illustrate the varieties.]



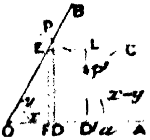
(c).

4. From *Fig.* 11, Art. 48, demonstrate geometrically the formula $\cos(x + y) = \cos x \cos y - \sin x \sin y$. The same for each of the cases in Ex's 1, 2, and 3, above.

Sug. In *Fig.* 11, $\cos(x + y) =$ OD′ = OF — LE. $\frac{\text{OF}}{\text{OE}} = \frac{\text{OD}}{\text{OP}'}$, or OF $= \cos x \cos y$. $\frac{\text{LE}}{\text{P}'\text{E}} = \frac{\text{PD}}{\text{OP}'}$, or LE $= \sin x \sin y$.

5. Prove geometrically the relation $\sin(x - y) = \sin x \cos y - \cos x \sin y$.

Sug. Let aP $= x$, and since y is to be subtracted we measure it *back* from P, and $y =$ PP′. Now $\sin(x - y) =$ P′D′ = EF — P′L.



6. Prove from *Fig.* (*d*) that $\cos(x - y) = \cos x \cos y + \sin x \sin y$.

7. Given $\sin 45° = \sqrt{\frac{1}{2}}$, and $\sin 30° = \frac{1}{2}$ to find sin 75°, and sin 15° Also tan, and cot. *Result*, $\sin 75° = .97$, $\sin 15° = .26$, nearly.

Sug. Use *formulæ* (A . . . . H).

8. Given $\sin 30° = \frac{1}{2}$, to find sine, cosine, tangent, and cotangent, of 15°, 7° 30′, 3° 45′, and 1° 52′ 30″. *Results*, $\sin 15° = .2588$, $\cos 15° = .97$, nearly.

Sug. Use the *formulæ* in (*57*). Compare results with those found in the Table of Natural Sines, etc.

9. Of what angles may the trigonometrical functions be found from $\sin 45° = \frac{1}{2}\sqrt{2}$, by means of the *formulæ* in (*57*)? How?

10. Prove that $\sin(x+y+z) = \sin x \cos y \cos z + \cos x \sin y \cos z + \cos x \cos y \sin z - \sin x \sin y \sin z$. Also, $\cos(x+y+z) = \cos x \cos y \cos z - \sin x \sin y \cos z - \sin x \cos y \sin z - \cos x \sin y \sin z$. Also that $\tan(x+y+z) = \dfrac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - \tan x \tan y - \tan x \tan z - \tan y \tan z}$.

SUG. $\operatorname{Sin}(x+y+z) = \sin[(x+y)+z] = \sin(x+y)\cos z + \cos(x+y)\sin z$.

SCH. Since if $(x+y+z) = \pi$, $\tan(x+y+z) = 0$, we have from the last form, $\tan x + \tan y + \tan z = \tan x \tan y \tan z$; *i. e.*, if a semicircumference be divided into any three parts, the sum of the tangents of the three parts equals the products of the tangents of the same.

11. Prove that $\sec(x+y) = \dfrac{\sec x \sec y \operatorname{cosec} x \operatorname{cosec} y}{\operatorname{cosec} x \operatorname{cosec} y - \sec x \sec y}$.

12. Prove that $\sin 3x = 3\sin x - 4\sin^3 x$. Also that $\cos 3x = 4\cos^3 x - 3\cos x$. Also, $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$. Also, $\cot 3x =$

$\operatorname{Sin} 3x = \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x = 2\sin x \cos x \cos x + (1 - 2\sin^2 x)\sin x = 2\sin x \cos^2 x + \sin x - 2\sin^3 x = 2\sin x\,(1 - \sin^2 x) + \sin x - 2\sin^3 x = 3\sin x - 4\sin^3 x$.

$\operatorname{Tan} 3x = \tan(2x + x) = \dfrac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$. In the latter substitute for $\tan 2x$ its value in terms of $\tan x$.

13. Prove that $\sin 4x = 4(\sin x - 2\sin^3 x)\cos x$.

14. Prove that $\sin x = \dfrac{2\tan\frac{1}{2}x}{1 + \tan^2\frac{1}{2}x} = \dfrac{2}{\cot\frac{1}{2}x + \tan\frac{1}{2}x}$.

SUG. $\operatorname{Sin} x = 2\sin\frac{1}{2}x \cos\frac{1}{2}x = \dfrac{2\sin\frac{1}{2}x}{\sec\frac{1}{2}x} = \dfrac{\frac{2\sin\frac{1}{2}x}{\cos\frac{1}{2}x}}{\sec\frac{1}{2}x} = \dfrac{2\tan\frac{1}{2}x}{\sec^2\frac{1}{2}x} = \dfrac{2\tan\frac{1}{2}x}{1 + \tan^2\frac{1}{2}x}$. To produce the last form divide numerator and denominator of $\dfrac{2\tan\frac{1}{2}x}{1 + \tan^2\frac{1}{2}x}$ by

15. Prove that $\tan\frac{1}{2}x = \dfrac{1 - \cos x}{\sin x}$. Also $\cot\frac{1}{2}x = \dfrac{1 + \cos x}{\sin x}$

SUG. From (L), (56), $2\sin^2\frac{1}{2}x = 1 - \cos x$, and from (K), $2\sin\frac{1}{2}x \cos\frac{1}{2}x = \sin x$. Divide the former by the latter.

16. Find the trigonometrical functions of 18°.

Solution.—Letting $x = 18°$, $2x = 36°$, and $3x = 54°$, hence $\sin 2x = \cos 3x$. But $\sin 2x = 2\sin x \cos x$; and $\cos 3x = 4\cos^3 x - 3\cos x$; hence $2\sin x \cos x = 4\cos^3 x - 3\cos x$, or $2\sin x = 4\cos^2 x - 3 = 4 - 4\sin^2 x - 3$. From which $4\sin^2 x + 2\sin x = 1$. Solving this quadratic, we have $\sin x$, or $\sin 18° = \dfrac{\sqrt{5} - 1}{4}$, neglecting the − root, since sin 18° is +. From this, $\cos 18° = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}$. These may be put in approximate decimal fractions.

17. Having given the functions of 18°, and 15° (Ex. 8), find those of 3°; then of 6°, 12°, 24°, etc.

Compare the results obtained with the values as given in the Table of Natural Sines, etc., obtaining all the values in decimal fractions.

SECTION III.

FORMULÆ FOR RENDERING CALCULABLE BY LOGARITHMS THE ALGEBRAIC SUM OF TRIGONOMETRICAL FUNCTIONS.

58. Since multiplication, division, involution, and evolution are the only elementary combinations of number which we can effect by means of logarithms, if we wish to add or subtract trigonometrical (or other) quantities, we have first to discover what products, quotients, powers, or roots, are equivalent to the proposed sums or differences.

59. Prop.—*To render* $\sin x \pm \sin y$, *and* $\cos x \pm \cos y$ *calculable by logarithms.*

Solution. From (*55*, Sch 2) we have $\sin(x + y) = \sin x \cos y + \cos x \sin y$, and $\sin(x - y) = \sin x \cos y - \cos x \sin y$. Adding these formulas $\sin(x + y) + \sin(x - y) = 2\sin x \cos y$. Now putting $x + y = x'$, and $x - y = y'$, whence $x = \frac{1}{2}(x' + y')$, and $y = \frac{1}{2}(x' - y')$; we have $\sin x' + \sin y' = 2\sin \frac{1}{2}(x' + y') \cos \frac{1}{2}(x' - y')$; or, dropping the accents, as the results are general,

$$\text{(A')} \qquad \sin x + \sin y = 2\sin \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y).$$

Again, by subtracting formula B (*55*, Sch. 2) from formula A, and making the same substitutions, we have,

$$\text{(B')} \qquad \sin x - \sin y = 2\cos \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y).$$

In like manner adding $\cos(x + y) = \cos x \cos y - \sin x \sin y$, and $\cos(x - y) = \cos x \cos y + \sin x \sin y$, and making the same substitutions, we have,

(C′) $$\cos x + \cos y = 2\cos \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y).$$

Finally subtracting formula C (*55*, Sch. 2) from formula D, and making the same substitutions, we have,

(D′) $$\cos y - \cos x = \quad 2\sin \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y), \text{ or}$$
$$\cos x - \cos y = -2\sin \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y).$$

60. Cor. 1.—*The sum of the sines of two angles is to their difference, as the tangent of one-half the sum of the angles is to the tangent of one-half their difference.*

Dem.—Dividing A′ by B′, we have,

$$\frac{\sin x + \sin y}{\sin x - \sin y} = \frac{\sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)}{\cos \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)} = \frac{\sin \frac{1}{2}(x + y)}{\cos \frac{1}{2}(x + y)} \times \frac{\cos \frac{1}{2}(x - y)}{\sin \frac{1}{2}(x - y)} =$$
$$\tan \tfrac{1}{2}(x + y) \cot \tfrac{1}{2}(x - y) = \tan \tfrac{1}{2}(x + y) \times \frac{1}{\tan \frac{1}{2}(x - y)} = \frac{\tan \frac{1}{2}(x + y)}{\tan \frac{1}{2}(x - y)}.$$ Q. E. D.

61. Cor. 2.—*The difference of the cosines of two angles divided by their sum is numerically equal to the product of the tangent of one-half the sum of the two angles into the tangent of one-half their difference.*

Dem.—Dividing D′ by C′, we have,

$$\frac{\cos x - \cos y}{\cos x + \cos y} = \frac{-\sin \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)}{\cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)} = -\frac{\sin \frac{1}{2}(x + y)}{\cos \frac{1}{2}(x + y)} \times \frac{\sin \frac{1}{2}(x - y)}{\cos \frac{1}{2}(x - y)} = -$$
$$\tan \tfrac{1}{2}(x + y) \tan \tfrac{1}{2}(x - y).$$ Q. E. D. (Observe the opposition in signs.)

---

62. Prob.—*To render* $\tan x \pm \tan y$ *calculable by logarithms.*

Dem.—$$\tan x \pm \tan y = \frac{\sin x}{\cos x} \pm \frac{\sin y}{\cos y} = \frac{\sin x \cos y \pm \cos x \sin y}{\cos x \cos y} = \frac{\sin (x \pm y)}{\cos x \cos y}$$ Q. E. D

EXERCISES.

Let the student deduce the following relations:

1. $\cot x + \cot y = \dfrac{\sin (x + y)}{\sin x \sin y}.$

2. $\sec x + \sec y = \dfrac{2 \cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)}{\cos x \cos y}.$

3. $\sec x - \sec y = \dfrac{2 \sin \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)}{\cos x \cos y}.$

4. $1 + \cos x = 2\cos^2 \frac{1}{2}x.$ (See **57.**)

5. $1 - \cos x = 2\sin^2 \frac{1}{2}x.$

6. $\frac{\sin x + \sin y}{\cos x + \cos y} = \tan \frac{1}{2}(x + y).$ (Divide A′ by C′, **59.**)

7. $\frac{\sin x - \sin y}{\cos x + \cos y} = \tan \frac{1}{2}(x - y).$

8. $\frac{\sin x + \sin y}{\cos x - \cos y} = -\cot \frac{1}{2}(x - y).$

9. $\frac{\sin x - \sin y}{\cos x - \cos y} = -\cot \frac{1}{2}(x + y).$

SECTION IV.

CONSTRUCTION AND USE OF TRIGONOMETRICAL TABLES.

[NOTE.—In order to read this and the subsequent sections, the student needs a knowledge of the nature of logarithms, and the method of using common logarithmic tables. If he is familiar with the last chapter in THE COMPLETE SCHOOL ALGEBRA of this series, he is prepared to go on. If he has not this knowledge, he should read the introduction preceding the table of Logarithms before reading this section.]

63.* A *Table of Trigonometrical Functions is a table containing the values of these functions corresponding to angles of all different values. In consequence of the incommensurability of an arc and its functions, these results can be given only approximately; yet it is possible to attain any degree of accuracy which practical science requires.

64. There are two tables of trigonometrical functions in common use, the *Table of Natural Functions*, and the *Table of Logarithmic Functions.*

65.* A *Table of Natural Trigonometrical Functions is a table in which are written the values of these functions for angles of various values, the radius of the circle being taken as the measuring unit, and the function being expressed in natural numbers extended to as many decimal places as the proposed degree of accuracy requires.

66.* A *Table of Logarithmic Trigonometrical Functions is the same as a table of natural functions, except that the *logarithms* of the values of the functions are written instead of the functions themselves, and to avoid the frequent occurrence of negative characteristics, the characteristic of each logarithm is increased by 10. For example, sines and cosines being always less than unity, except at the limit (*33*), and tangents of angles less than 45° and cotangents of angles greater than 45° being also less than unity, the logarithms of all such functions have negative characteristics. To obviate the necessity of writing these with their sign, the characteristic of each logarithm is increased by 10.

---

67. Prob.—*To compute a table of natural trigonometrical functions for every degree and minute of the quadrant.*

SOLUTION.—It is evident that an arc is longer than its sine, but that this disparity diminishes as the arc grows less. Thus, in a circle whose radius is 1 inch, the length of the sine of an arc of 1° would not differ appreciably from the arc. Much less should we be able to distinguish between the sine of 1′ and the arc. Now, since when the radius is 1, a semicircumference $= \pi = 3.1415926$, and also $= 180°$, or $180 \times 60 = 10800′$, we have the length of an arc of $1' = \frac{3.1415926}{10800} = 0.0002908882$ approximately. Assuming this as the sine of 1′, we obtain the cosine thus,

$\cos 1' = \sqrt{1 - \sin^2 1'} = \sqrt{(1 + \sin 1') \times (1 - \sin 1')} = \sqrt{1.0002908882 \times .9997091118}$ $= 0.9999999577$.

Having thus obtained sufficiently accurate values of sin 1′ and cos 1′, we can continue the operation as follows: from the *formulæ* $\sin(x + y) + \sin(x - y) = 2 \sin x \cos y$, and $\cos(x + y) + \cos(x - y) = 2 \cos x \cos y$, we have

$$\sin(x + y) = 2 \sin x \cos y - \sin(x - y),$$
$$\cos(x + y) = 2 \cos x \cos y - \cos(x - y).$$

Now letting y remain constantly equal to 1′, and letting x take successively the values 1′, 2′, 3′, etc., we have

$$\text{For } x = 1', \begin{cases} \sin 2' = 2 \cos 1' \sin 1' - \sin 0' = 0.0005817764 \\ \cos 2' = 2 \cos 1' \cos 1' - \cos 0' = 0.9999998308 \end{cases}$$

$$\text{For } x = 2', \begin{cases} \sin 3' = 2 \cos 1' \sin 2' - \sin 1' = 0.0008726646 \\ \cos 3' = 2 \cos 1' \cos 2' - \cos 1' = 0.9999996193 \end{cases}$$

$$\text{For } x = 3', \begin{cases} \sin 4' = 2 \cos 1' \sin 3' - \sin 2' = 0.0011635526 \\ \cos 4' = 2 \cos 1' \cos 3' - \cos 2' = 0.9999993232 \end{cases}$$

$$\text{For } x = 4', \begin{cases} \sin 5' = 2 \cos 1' \sin 4' - \sin 3' = 0.0014544407 \\ \cos 5' = 2 \cos 1' \cos 4' - \cos 3' = 0.9999989425 \end{cases}$$

etc., etc.

These operations present no difficulties except the labor of performing the numerical operations.

Of course 60 operations are required for every degree, and for 30°, 1800. But having computed the sines and cosines for every degree and minute up to 30°

we can complete the work by simple subtraction of values already found. For example, letting $x = 30°$, the first formula used above becomes

$$\sin (30° + y) = \cos y - \sin (30° - y),$$

and from $\cos (x + y) - \cos (x - y) = -2 \sin x \sin y$, we have

$$\cos (30 + y) = \cos (30° - y) - \sin y.$$

Now making y successively $= 1', 2', 3'$, etc., these give

$$\begin{cases} \sin 30°\,1' = \cos 1' - \sin 29°\,59' \\ \cos 30°\,1' = \cos 29°\,59' - \sin 1' \end{cases}$$

$$\begin{cases} \sin 30°\,2' = \cos 2' - \sin 29°\,58' \\ \cos 30°\,2' = \cos 29°\,58' - \sin 2' \end{cases}$$

$$\begin{cases} \sin 30°\,3' = \cos 3' - \sin 29°\,57' \\ \cos 30°\,3' = \cos 29°\,57' - \sin 3' \end{cases}$$

etc., etc.

All of these values which occur in the second members having been determined in reaching sin 30° and cos 30°, those in the first members can be found by performing the requisite subtractions.

Proceeding in this way till we reach 45°, the numerical values of *all* sines and cosines become known, since the sine of any angle between 45° and 90°, being the cosine of the complementary angle, will have been computed in reaching 45°. And so also the cosines of angles between 45° and 90° will have been computed as sines of the complementary angles below 45°.

The sines and cosines being computed, the corresponding tangents, cotangents, and, if need be, the secants, cosecants, versed-sines, and coversed-sines, can be calculated from the relations $\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{1}{\tan x}$ or $\frac{\cos x}{\sin x}$, $\sec x = \frac{1}{\cos x}$, $\operatorname{cosec} x = \frac{1}{\sin x}$, $\operatorname{vers} x = 1 - \cos x$, and $\operatorname{covers} x = 1 - \sin x$.

68. SCH.—If it is desired to obtain the natural functions of angles estimated to seconds, it is necessary that the values in the tables computed as above be extended to 7 decimals at least. From such a table we may make interpolations for seconds with sufficient accuracy for most practical ends, except for values near the limits, where the disparity between the variation of the arc and that of the function changes very rapidly. For example, let it be required to find sin 34° 24′ 12″ from the data sin 34° 24′ = .5649670, and sin 34° 25′ = .5652070. We observe that an increase of 1′ upon the angle of 34° 24′ makes an increase of .5652070 − .5649670 = .0002400 in the sine. Hence an increase of 12″, or $\frac{1}{5}$ of 1′, makes an increase of $\frac{1}{5}$ of .0002400, or .0000480, *approximately*. Adding, we have sin 34° 24′ 12″ = .5650150. The student must be careful to notice whether an increase of the angle makes a numerical increase or a decrease of the function, and add *or subtract* as the case may require.

69. Prob.—To construct a table of logarithmic trigonometrical functions.

SOLUTION.—Compute the natural sines and cosines as in the preceding problem. Take the logarithms of the values thus obtained, and add 10 to each

characteristic. The results are the ordinary tabular logarithmic sines and cosines. For example, we find from the table of natural functions that sin 34° 24′ = .5649670. The logarithm of this number is $\bar{1}.752023$. Adding 10 to the characteristic, we have log sin 34° 25′ = 9.752023, as usually given in the tables. In like manner the cosines are obtained.

To obtain the tabular logarithmic tangents, we have from $\tan x = \frac{\sin x}{\cos x}$ $\log \tan x = \log \sin x - \log \cos x$. If we now take the log sin x from the table as computed by the preceding part of this solution, and from it subtract the corresponding log cos x, the result is the *true* log tan x, since the extra 10 in the tabular log sin and log cos is destroyed by the subtraction. Therefore, to this difference we must add 10 to get the tabular log tan, as above explained. For example, the tabular log sin 34° 24′ = 9.752023, and log cos 34° 24′ = 9.916514. Hence, the tabular log tan 34° 24′ = 9.752023 − 9.916514 + 10 = 9.835509. In like manner the tabular log cot x = log cos x − log sin x + 10.

If the logarithmic secants are required they can be obtained from the relation $\sec x = \frac{1}{\cos x}$, which gives $\log \sec x = 0 - \log \cos x$. In applying this by means of the tabular functions, it must be observed that the log cos x, as we get it from the table, is 10 too great; hence, the *true* $\log \sec x = 0 - \log \cos x + 10$. In tabulating log secants and cosecants, it is not necessary to add 10, since, as these functions are never less than 1, their logarithms are never negative.

70. SCH.—The interpolations for seconds are usually made in the same way when using the logarithmic functions, as explained above for the natural functions. But to facilitate the operation, the approximate change of the logarithm for a change of 1″ of the angle is commonly written in the table, in a column called *Tabular Differences*, and marked D.

EXERCISES.

1. Find from the tables at the close of the volume the natural trigonometrical functions of 25° 18′.

SOLUTION.—*To find the sine and cosine* we look in Table II., and find 25° at the top of the page. In the extreme left-hand column we find the minutes, and passing down to 18, find opposite, in the column headed N. sin (natural sine) 42736; also in the column N. cos, we find 90408. Now, as these are the lengths of the sine and cosine as compared with radius, we know they are fractions. ∴ Sin 25° 18′ = .42736, and cos 25° 18′ = .90408.

To find the tangent we turn to Table IV., and finding 25° at the top of the page, pass down the column of minutes, on the left-hand of the page, to 18, opposite which, and under the column headed 25°, we find 2698. To this we prefix the figures 47, which stand in the same column, opposite 11′, and belong to the tangents of all the angles from 25° 10′ to 25° 19′, and are omitted in the table simply to relieve the eye and to economize space. Thus we find tan 25° 18′ = .472698, the number being known to be a fraction because the angle is less than 45°

To find the cotangent we look at the *bottom* of the page in the same table till we find 25°, and then passing *up* the minutes column at the *right* hand, find cot 25° 18′ = 2.11552.

If the secant were required we should be obliged to obtain it by dividing 1 by the cosine, as our tables do not include this function. Thus sec 25° 18′ =

$$\frac{1}{\cos 25° 18'} = \frac{1}{.90408} = 1.1061.$$

[NOTE.—Tables of secants and cosecants are sometimes given, but they are not of sufficient importance to justify their introduction into an elementary text-book.]

2. Show that sin 37° 43′ = .61176; cos 37° 43′ = .79105; tan 37° 43′ = .773353; cot 37° 43′ = 1.29307; sec 37° 43′ = 1.264142; cosec 37° 43′ = 1.634628; vers 37° 43′ = .20895; covers 37° 43′ = .38824.

3. Find that sin 64° 36′ = .90334; cos 64° 36′ = .42894; tan 64° 36′ = 2.10600; cot 64° 36′ = .474835; sec 64° 36′ = 2.331328; cosec 64° 36′ = 1.107003; vers 64° 36′ = .57106; covers 64° 36′ = .09666.

SUG.—In looking for sines and cosines of angles above 45°, seek the degrees at the *bottom* of the page, and be careful to observe that the columns of sines and cosines, as named at the top, change names when read from the bottom. The foundation of this arrangement will be readily perceived. Thus, turning in Table II. to 24° 32′, we find sin 24° 32′ = .41522. But sin 24° 32′ = cos (90 − 24° 32′) = cos 65° 28′ = .41522. Thus the degrees and minutes read from the bottom of the page are the complements of those read from the top.

4. Find that sin 42° 27′ 12″ = .67499; cos 42° 27′ 12″ = .73783; tan 42° 27′ 12″ = .914834; cot 42° 27′ 12″ = 1.09309.

SUG.—Sin 42° 27′ = .67495, and sin 42° 28′ = .67516. ∴ An increase of 1′ in the angle makes an increase of 21 (hundred thousandths) in the sine, and 12″ will make $\frac{12}{60}$ or $\frac{1}{5}$ as great an increase, approximately. Observe that in the case of cosine an increase of the arc makes a *decrease* of the function.

5. Find that sin 143° 24′ = 0.596225; cos 151° 23′ = .877844; tan 132° 36′ = 1.08749; and cot 116° 7′ = .490256.

SUG.—Sin 143° 24′ = sin (180° − 143° 24′) = sin 36° 36′. Also the trigonometrical function of any angle is numerically equal to the same function of its supplement (*56*).

6. Find the logarithmic trigonometrical functions of 32° 15′ 22″ from the tables at the end of the volume.

SOLUTION.—Turning to Table II. we find 32° at the top of the page, and opposite 15′, and in the column L. sin (logarithmic sine), we get 9.727228; *i. e.*, log sin 32° 15′ = 9.727228. Now from the column of differences, D. 1″, we learn that an increase of 1″ of the arc at this point makes, approximately, an increase of 3.34 (millionths) in the logarithm of its sine. Hence, we *assume* that an increase of 22″ makes 22 × 3.34 = 73 (millionths). ∴ log sin 32° 15′ 22″ = 9.727228 + .000073 = 9.727301. In a similar manner we have log cos 32° 15′ = 9.927231. An increase of 1″ in the arc makes a *decrease* of 1.33 (millionths) in the log cos. ∴ an increase of 22″ makes 29 (millionths) *decrease* in the log cos, and log cos 32° 15′ 22″ = 9.927202. Log tan 32° 15′ 22″ = 9.800100; and log cot 32° 15′ 22″ = 10.199901.

7. Find that log sin 24° 27′ 34″ = 9.617051; log cos 26° 12′ 20″ = 9.952897; log tan 26° 12′ 20″ = 9.692125; log cot 126° 23′ 50″ = 9.867579.

SUG.—Observe cot (126° 23′ 50″) = cot (180° − 126° 23′ 50″) = cot (53° 36′ 10″). Also that angles above 45° are found at the bottom of the table; and remember to subtract the correction for co-functions, if an increase of arc is assumed.

---

8. Given the natural sine .45621, to find the angle from the tables.

SOLUTION.—Looking for this sine in the table of natural sines, we find the next less sine to be .45606, and the angle corresponding, 27° 8′. Now, at this point, an increase of 1′ in the arc makes an *increase* of 26 (hundred thousandths) in the natural sine. But the given sine .45621 is only 15 (hundred thousandths) greater than .45606, the sine of 27° 8′. Hence the required angle is but $\frac{15}{26}$ of 1′ or 60″ = 35″, greater than 27° 8′. ∴ $\sin^{-1}.45621$ = 27° 8′ 35″, and its supplement 152° 51′ 25″, which has the same sign, and these arcs increased by every multiple of 2π.

9. Find $\sin^{-1}.62583$; $\cos^{-1}.34268$; $\tan^{-1}.468531$; $\cot^{-1}.876434$.

Results. $\text{Sin}^{-1}.62583$ = 38° 44′ 35″, and 141° 15′ 25″; and these arcs increased by every multiple of 2π.

$\cos^{-1}.34268$ = 69° 57′ 36″, and 360° − 69° 57′ 36″ = 290° 2′ 24″, and these arcs increased by every multiple of 2π.

$\tan^{-1}.468531$ = 25° 6′ 16″, and 180° + 25° 6′ 16″ = 205° 06′ 16″, and these arcs increased by every multiple of 2π.

$\cot^{-1}.876434$ = 48° 46′ 3″, and 180° + 48° 46′ 3″ = 228° 46′ 03″, and these arcs increased by every multiple of 2π.

SUG.—Observe that an *increase* of the arc makes a *decrease* of its co-functions. In the table of tangents as given, Table IV., the proportional parts given at the bottom of each column are the approximate changes which the functions undergo for a change of 1″ in the function. Thus, in finding $\cot^{-1} .876434$, we find $\cot^{-1} .876462 = 48° 46'$; and at the bottom we find that a change of 8.64 (millionths) in the function makes a change of 1″ in the angle. Hence, as the given cotangent is 28 (millionths) *less* than the cotangent of 48° 46′, the angle required is $28 \div 8.64 = 3$ (seconds), *greater* than 48° 46′.

71. SCH.—It is usually best to take from the table that function which is nearest in value to the given function, and then increase or diminish the corresponding arc as the case may require. If we always take from the table the next *less* function than that given in the example for sine, tangent, and secant, and the next *greater* for the cosine, cotangent, and cosecant, corrections for seconds will require always to be *added*. If we always take from the tables the functions *next less* than the one given, the corrections for seconds must be *added* for sine, tangent, and secant, and *subtracted* for the co-functions. If we were always to take from the tables the next *greater* function than the one given, the seconds corrections would be *added* for the co-functions, and *subtracted* for the others.

[NOTE.—It is very important that the pupil become so familiar with the nature of these tables as to use them intelligently, and not mechanically. For this reason we refrain from giving the usual specific, mechanical directions for their use, and substitute illustrations showing how they are used in accordance with the principles upon which they are constructed.]

10. Find $\sin^{-1}(-.34256)$; $\cos^{-1}(-.62584)$; $\tan^{-1}(-3.41621)$; $\cot^{-1}(-1.21648)$.

Results. $\sin^{-1}(-.34256) = 200° 1' 58''$, and $339° 58' 02''$, and these arcs increased by every multiple of 2π.

$\cos^{-1}(-.62584) = 128° 44' 38''$, and $231° 15' 22''$, and these arcs increased by every multiple of 2π.

$\tan^{-1}(-3.41621) = 106° 18' 57''$, and $286° 18' 57''$, and these arcs increased by every multiple of 2π.

$\cot^{-1}(-1.21648) = 140° 34' 42''$, and $320° 34' 42''$, and these arcs increased by every multiple of 2π.

SUG.—To obtain these results the pupil will need to recall the principles in the corollaries to (*48—55*). Thus, to find $\cot^{-1}(-1.21648)$, we find from the table that $\cot^{-1}(1.21648) = 39° 25' 18''$; and from (*55*) Cor., we learn that $\cot(180° - x) = -\cot x$. $\therefore \cot^{-1}(-1.21648) = 180° - 39° 25' 18'' = 140° 34' 42''$. Again, from the same corollary, we learn that $\cot(300° - x) = -\cot x$. $\therefore \cot^{-1}(-1.21468) = 360° - 39° 25' 18'' = 320° 34' 42''$.

---

11. Given the logarithmic sine 9.451234, to find the corresponding angle.

Solution.—The next nearest log sin found in Table II., is 9.451204 = log sin 16° 25′. Now we learn from the table that an increase of 1″ in the angle at this point, makes an *increase* in its log sin of 7.14 (millionths). But the given log sin, 9.451234, is 30 (millionths) greater than log sin 16° 25′. ∴ The required angle is 30 ÷ 7.14 = 4 (seconds) greater than 16° 25′; and we have sin 16° 25′ 4″ = 9.451234. Again, as sin 16° 25′ 4″ = sin (180° − 16° 25′ 4″) = sin 163° 34′ 56″ the latter angle has for its log sin 9.451234. Finally, either of these angles increased by any multiple of 2π has the same logarithmic sine.

12. Show from the table that the angle whose log cos is 9.778151, is 53° 7′ 49″, and also 306° 52′ 11″, and each of these angles increased by any multiple of 2π.

13. What angles correspond to the logarithmic cosines 9.246831 and 9.889372?

14. Find from the table what angles have for their logarithmic tangents 9.895760, 10.531054, and 11.216313.

Results. The first two are the log tans of 38° 11′ 20″, and 73° 35′ 43″, and also of 180° + either of these angles, and each increased by any multiple of 2π.

15. Find the angles corresponding to the logarithmic cotangents 10.008688, 9.638336, and 9.436811.

Results. The first two are the log cots of 44° 25′ 37″, and 66° 29′ 54″, and also of 180° + either of these angles, and each increased by any multiple of 2π.

---

72. Sch.—Strictly speaking, negative numbers have no logarithms; since no base can be assumed, such that all negative numbers can be represented by said base affected with exponents. It is therefore customary to say that negative numbers have *no* logarithms. Nevertheless, *we do apply logarithms to negative trigonometrical functions.* Thus, if we have $-\cos x$, the $-$ sign is interpreted as simply telling in what quadrants x may end; while, in other respects, the function is treated exactly like $+\cos x$.

16. Given $\log(-\cos x) = 9.346251$, to find x.

Solution.—The logarithmic cosine 9.346261, considered independently of its sign, corresponds to 77° 10′ 35″. But the $-$ sign requires that the arc shall end in the 2d or 3d quadrant, for such angles, and such only, have negative cosines. ∴ The angles required are 180° ∓ 77° 10′ 35″ = 102° 49′ 25″, and 257° 10′ 35″, and these increased by entire circumferences, as all these angles have logarithmic cosines, which are numerically equal to 9.346261, and the cosines themselves are negative.

17. What angle less than 180° has a negative cosine whose tabular logarithmic value is 9.653825? *Ans.* 116° 47′ 4″

18. What angle less than 180° has a negative tangent whose tab lar logarithmic value is 9.884130? *Ans.* 142° 33′ 15

19. What angle less than 180° has a negative sine whose tu logarithmic value is 9.341627?

20. What angle less than 180° has a negative co tabular logarithmic value is 9.564299? *An*

21. Find values of $x < 180°$ which fulfil the following co

$$\log(-\cos x) = 9.562468;\ \log(-\tan x) = 10.764215;$$
$$\log(-\sin x) = 8.886432;\ \log(-\cot x) = 11.152161.$$

Results. 111° 25′; 99° 45′ 54″; none; 175° 58′ 14″.

22. Having at hand only the common logarithmic tables of trigonometrical functions, and the table of logarithms of numbers, I wish to find the number of degrees, minutes, and seconds corresponding to the natural tangent 2.16145. How is it done, and what is the result?

Answer: Find the logarithm of 2.16145, to this add 10, and find the angle corresponding to this tabular logarithmic tangent. The angle is 65° 10′ 20″.

23. From the same tables as above find the natural cosine of 35° 23′. Also what angle corresponds to natural tangent 2.

24. From the same tables as above find the angle corresponding to natural tangent − 1.82645. Also to natural cosine − .42536.

25. Why is it in the table of logarithmic functions that the sine of an angle minus its cosine + 10 gives the tangent? Why that cosine − the sine + 10 gives the cotangent? Why that the sum of the tangent and cotangent of any angle = 20? Why is but one column of tabular differences needed for tangents and cotangents, while the sines and cosines require each a separate column?

FUNCTIONS OF ANGLES NEAR THE LIMITS OF THE QUADRANT.

TABLE III.

[NOTE.—This subject may be omitted in an elementary course, the first time going over, if thought best.]

73. ***Failure of Table II.***—The method which has been given in the preceding pages for finding the logarithmic functions of angles involving seconds, by means of the *Tabular Differences*, Table II., is sufficiently accurate in most cases for practical purposes, but is tirely too rude for the sines, tangents, and cotangents of angles ar the beginning of the quadrant (those less than 2° or 3°), and for cosines, tangents, and cotangents of angles near the close of the quadrant (those between 87° or 88° and 90°). An example will render this clear. Suppose we wish to find log sin 1′ 12″. We find from Table II., $\log \sin 1' = 6.463726$; and also that the *average* increase of the log sin between 1′ and 2′ is 5017.17 (millionths) for every second increase of the angle. But this *average* rate of increase of the function during the minute is much *less* than its *real* rate of increase *in the first part of the minute*, as from 1′ to 1′ 12″, and much *greater* than the real rate of increase *in the latter part*, as the angle approaches 2′. In fact, we see from this table, that we should use 2934.85 as the increase of log sin 2′ for 1″ increase of the arc. Now, in our proposed example, we want the increase of the log sin while the angle is passing from 1′ to 1′ 12″. This, as shown above, is considerably *more* than 5017.17 (millionths) for every second.

The cosine being the sine of the complement is subject to the same law of change near the *close* of the quadrant, that governs the sine at the beginning.

The case of the *tangent* of a small angle is similar to that of the sine; and since the cotangent is the reciprocal of the tangent, it has the same *law* of change, only that the one increases as the other decreases. Thus, since doubling a small arc, as 1″, doubles its tangent (approximately), it divides its cotangent by 2.

Finally, while the law of change in the sine is very different near the close of the quadrant from what it is near the beginning, the sine changing very rapidly at the beginning and very slowly at the close, and the cosine is just the opposite, the tangent, and cotangent have the same law of change at both extremities of the quadrant. Thus, if near the beginning of the quadrant a certain small increase of the arc *increases* the tangent at a particular rate, it *decreases* the cotangent *at the same rate*, since these functions are reciprocals of each other. Moreover, since $\tan 1° = \cot (90° - 1°) = \cot 89°$, cot 89° changes according to the same law as tan 1°; and tan 89 changes reciprocally with cot 89°.

74. Description of Table III.—The first page of the table enables us to find the sines of angles less than 2° 36′ 15″ (and consequently the cosines of angles between 87° 23′ 45″ and 90°) with very great accuracy. The columns headed *Angles* contain the degrees, minutes, and seconds of the proposed angles, and the columns at their right give the same angles in seconds. The columns headed

Diff. contain the corrections to be used according to the following problems. The second and third pages answer a similar purpose with reference to tangents and cotangents of arcs within 2° 36′ 20″ of the limits of the quadrant.

---

75. Prop.—*Letting x represent any number of seconds less than 2° 36′ 15″, we have,*

$$\log \sin x'' = 4.685575 + \log x - \text{Diff.}$$

Dem.—The length of 1″ of an arc to radius unity is 3.14159265358979 (the length of the semicircumference) ÷ 648000 (the number of seconds in 180°), and = .00000484812. For practical purposes this fraction may also be taken as the sine of 1″. Though, actually, the sine is less than the arc, the expressions for arc 1″ and sine 1″ agree to as many places of decimals as we have here. Again, for these small arcs the sine increases at *nearly* the same rate as the arc, so that $\sin 3'' = .00000484812 \times 3$ *nearly*; $\sin 102'' = .00000484812 \times 102$ *nearly*; these results being slightly in excess of the true values. It is the correction for this excess that is furnished by Table III in the columns marked *Diff.* But this table is adapted to logarithmic computation; hence we have $\log \sin x'' = \log \sin 1'' + \log x - \text{Diff.}$ In this expression $\log \sin x''$ is the logarithm of the natural sine of x'' (not increased by 10, as each function in Table II is); $\log \sin 1'' + \log x$, that is, the logarithmic sine of 1″ plus the logarithm of the number of seconds, corresponds to multiplying the sin 1″ by the number of seconds, and gives the logarithm of the product, or strictly, the logarithm of the length of the arc of x''. Now, the sine of x'' being less than the arc, its logarithm is less than the logarithm of the length of the arc. Just how much less the table tells. This difference, therefore, between the logarithm of the arc and the logarithm of its sine, which is given in the table, is to be subtracted. Finally, to make this result agree with Table II. we must add 10 to the result. Now, $\log \sin 1'' = \log .00000484812 = \bar{6}.685575$, and adding 10, we have 4.685575.

$$\therefore \log \sin x'' = 4.685575 + \log x - \text{Diff.},$$

a result which agrees with the logarithmic functions in Table II. Q. E. D.

76. Cor. 1.—*To obtain the log cos of an angle between 87° 23′ 45″ and 90°, from this table, take the log sin of its complement.*

77. Cor. 2.—*To obtain the log tan of an angle less than 2° 36′ 20″, from this table, use the formula,*

$$\log \tan x'' = 4.685575 + \log x + \text{Diff.}$$

Dem.—For as small an arc as 1″, sine, arc, and tangent are practically equal; hence, $\log \tan 1'' = \log \sin 1'' = 4.685575$ (10 being added). Moreover, for these small arcs the tangents increase (like the sines) in *nearly* the same ratio as the arcs; hence, we add $\log x$. Finally, the tangent is a little in excess of the arc, which excess is given in the table, and is to be *added*. Q. E. D.

78. Cor. 3.—*To obtain the log cot of an angle less than* 2° 36′ 15″, *from this table, use the formula,*

$$\log\cot x'' = 15.314425 - \log x - \text{Diff.}$$

Demonstration.—Since $\cot x'' = \dfrac{1}{\tan x''}$, $\log\cot x'' = \log 1 - \log\tan x'' = 20 - (4.685575 + \log x + \text{Diff.}) = 15.314425 - \log x - \text{Diff.}$ The 20 arises from adding 10 twice to log 1 (= 0). One 10 is added because 4.685575 is 10 in excess of the true log tan 1″; and the other 10 is added in order to make the log cot x'' agree with the ordinary tabulated value, as in Table II. Q. E. D.

79. Cor. 4.—*To obtain the log tan of an angle between* 87° 23′ 45″ *and* 90°, *take the log cot of its complement; and to obtain the log cot of an angle between the same values, take the log tan of its complement.*

---

80. ***Prob.***—*Having given a log sin less than* 8.657397 (*the log sin* 2° 36′ 15″), *to find the corresponding angle.*

Solution.—From $\log\sin x'' = 4.685575 + \log x - \text{Diff.}$, we have, $\log x = \log\sin x'' - 4.685575 + \text{Diff.}$ Hence, if from the given log sin, we subtract 4.685575, and then add the proper correction as furnished by Table III., we have the logarithm of the number of seconds sought. But we cannot tell what Diff. to take till we know the number of seconds. To meet this difficulty, find the angle corresponding to the given log sin from Table II., and reduce it to seconds. This will be sufficiently accurate to furnish the required Diff.

81. Cor. 1.—*Having given a log cos less than* 8.657397 (*the log cos of* 87° 23′ 45″), *to find the corresponding angle, treat it as if it were a log sin, and having found the corresponding angle, take its complement.*

82. Cor. 2.—*For log tan and log cot, the formulæ in* (***77***, ***78***), *give,*

$$\log x = \log\tan x'' - 4.685575 - \text{Diff.},$$
$$\text{and, } \log x = 15.314425 - \log\cot x'' - \text{Diff.}$$

These are applied as in (***80***); that is, the Diff. to be subtracted is found by getting from Table II. the required angle in seconds, as near as may be, and then take from Table III. the corresponding Diff.

EXAMPLES.

1. Find the log sin, tan, and cot of 1° 11′ 15″.

Solution.—Log sin $1° 11' 15'' = 4.685575 + \log 4275 - .000031 = 8.316480$. $1° 11' 15'' = 4275''$. Since 4275″ is between 4230″ and 4300, the Diff. is 31 (millionths).

$$\text{Log tan } 1° 11' 15'' = 4.685575 + \log 4275 + .000062 = 8.316573.$$

$$\text{Log cot } 1° 11' 15'' = 15.314425 - \log 4275 - .000062 = 11.683427.$$

Or, log cot can be found by subtracting log tan from 20.

2. Verify the following by using Table III.:

$\log \sin 56' 26'' = 8.215242$; $\log \tan 56' 26'' = 8.215301$;
$\log \cot 56' 26'' = 11.784699$.

3. Verify the following by using Table III.:

$\log \cos 88° 17' 44'' = 8.473396$; $\log \tan 88° 17' 44'' = 11.526412$;
$\log \cot 88° 17' 44'' = 8.473588$.

---

4. Having given the logarithmic sine 7.246481 to find the angle.

Solution.—From Table II. we find $6' 5'' = 365''$ as the angle. But this is subject to the inaccuracy exhibited in (*73*). To obtain the correct result from Table III., we have (*80*),

$$\log x = 7.246481 - 4.685575 + 0 = 2.560906.$$

$$\therefore x = 363.8;\ \text{or the angle is } 6' 3''.8.$$

5. Given the logarithmic tangent 7.805487, to find the corresponding angle.

Solution.—Table II. gives $21' 58''.3 = 1318''.3$ as the angle. From Table III. the Diff. corresponding to this is 6 (millionths). Hence,

$$\log x = \log \tan x'' - 4.685575 - \text{Diff. } (\mathit{82})$$

$$\text{becomes, } \log x = 7.805487 - 4.685575 - .000006 = 3.119906.$$

$$\therefore x = 1318;\ \text{and the angle is } 1318'' = 21' 58''.$$

6. Given the logarithmic cotangent 12.197148, to find the corresponding arc.

Table II. gives the angle 21′ 40″.8, *but the true angle as given by Table III. is* 21′ 50″.

SECTION V.

TRIGONOMETRICAL SOLUTION OF PLANE TRIANGLES.

83. There are six parts in every plane triangle: three sides and three angles; one side and any other two of which being given, the remaining parts can be found by means of the relations which exist between the sides and tabulated trigonometrical functions. To exhibit these highly important practical operations is the object of this section. We shall treat first of right angled plane triangles, and then of oblique angled plane triangles.

OF RIGHT ANGLED TRIANGLES.

Prop. 84.—*The relations between the sides and the trigonometrical functions of the oblique angles of a right angled triangle are as follows:*

$$(1)\ \text{sine} = \frac{\text{side opposite}}{\text{hypotenuse}}; \quad \therefore \text{cosecant} = \frac{\text{hypotenuse}}{\text{side opposite}};$$

$$(2)\ \text{cosine} = \frac{\text{side adjacent}}{\text{hypotenuse}}; \quad \therefore \text{secant} = \frac{\text{hypotenuse}}{\text{side adjacent}};$$

$$(3)\ \text{tangent} = \frac{\text{side opposite}}{\text{side adjacent}}; \quad \therefore \text{cotangent} = \frac{\text{side adjacent}}{\text{side opposite}}.$$

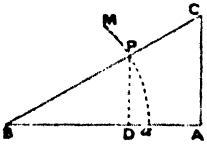


FIG. 12.

DEM.—Let CAB, *Fig.* 12, be a triangle, right angled at A. Let aM be the measuring arc of the angle B, PD = sin B, and BD = cos B. From the similar triangles PDB and CAB, we have $\frac{PD}{BP} = \frac{CA}{BC}$; *i. e.*, $\sin B = \frac{\text{side opposite}}{\text{hypotenuse}}$, since $BP = 1$. From the same triangles $\frac{BD}{BP} = \frac{BA}{BC}$; *i. e.*, $\cos B = \frac{\text{side adjacent}}{\text{hypotenuse}}$. Tangent being equal to sine divided by cosine, we have $\tan B = \frac{\text{side opposite}}{\text{hypotenuse}} \div \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{\text{side opposite}}{\text{side adjacent}}$. The other functions being the reciprocals of these three, are as given in the proposition. Finally, as a similar construction could be made about the other oblique angle, C, this demonstration may be considered general. Q. E. D.

SCH. 1.—These *formulæ* are so important that it is well to have them fixed in the memory, not only as written above, but also as follows:

$$\text{(1). Side-opp} = \text{hy} \times \sin, \text{ or } \frac{\text{hy}}{\text{cosec}}, \text{ or } \tan \times \text{side-adj}, \text{ or } \frac{\text{side-adj}}{\cot};$$

$$\text{(2). Side-adj} = \text{hy} \times \cos, \text{ or } \frac{\text{hy}}{\sec}, \text{ or } \cot \times \text{side-opp}, \text{ or } \frac{\text{side-opp}}{\tan};$$

of which the relations side-opp = hy × sin, and side-adj = hy × cos are of the most frequent use.

SCH. 2.—The six ratios given in this proposition are frequently made the definitions of the trigonometrical functions. Thus, referring to a right angled triangle, a sine of an angle may be defined to be the ratio of the side opposite to the hypotenuse; the cosine as the ratio of the side adjacent to the hypotenuse, etc.

SCH. 3.—The student will be aided in remembering these important relations by observing that the side opposite the angle is analogous to the sine, and the side adjacent to the cosine. Now, the sine $= \frac{\text{its analogue}}{\text{hy}}$, and so also the cosine. Tangent equals sine divided by cosine, and in this case it is the part analogous to the sine, divided by the part analogous to the cosine. One should not make the blunder of saying that $\sin = \frac{\text{hy}}{\text{side opp}}$, since that would make the sine always more than 1; but we have seen that it never can exceed 1. Similar checks against error may be made in the case of the other relations.

EXERCISES.

[NOTE.—The first five of these exercises are mainly designed to illustrate the proposition, and familiarize the mind with the relations.]

1. In a right angled triangle whose sides are 3, 4, and 5, what are the trigonometrical functions of the angles? What are the functions when the sides are 6, 8, and 10?

2. In a right angled triangle the hypotenuse is 12, and the angle at the base $\sin^{-1}\frac{1}{2}$. What are the sides? What is the sine of the other angle?

SUG.—Represent the angles by B, A, and C, A being the right angle; and the sides opposite by b, a, and c. Then $\sin B = \frac{b}{a}$, or $\frac{1}{2} = \frac{b}{12}$; whence, $b = 6$ $\sin C = \cos B = \frac{1}{2}\sqrt{3}$, $c = 6\sqrt{3}$.

3. In a right angled triangle whose hypotenuse is 12, and the angle at the base $\tan^{-1}2$, what are the other parts?

Ans. $\text{Cos}^{-1}\frac{1}{5}\sqrt{5}$, $\frac{12}{5}\sqrt{5}$, and $\frac{24}{5}\sqrt{5}$.

4. The sides of a right angled triangle are 20 and 32. What are the angles and hypotenuse? Obtain the hypotenuse by means of the secant.

Ans. $\operatorname{Tan}^{-1}\frac{5}{8}$, $\tan^{-1}\frac{8}{5}$, and $4\sqrt{89}$.

5. The hypotenuse of a right angled triangle is 120, and one side 100. Show that the angle opposite the latter is $\tan^{-1}\frac{5}{11}\sqrt{11}$, the adjacent angle $\operatorname{cosec}^{-1}\frac{6}{11}\sqrt{11}$, and the remaining side $20\sqrt{11}$. Obtain these results in the order given. Use a trigonometrical function to obtain the last.

EXAMPLES.

(*a*) BY MEANS OF THE TABLE OF NATURAL FUNCTIONS.

1. In a right angled triangle ABC, the hypotenuse BC is 235, and the angle B is 43° 25′. Find the angle C, and the sides AB and AC.

C = 46° 35′; AB = 170.7; AC = 161.52.

SOLUTION —*To find* C, we have but to remember that the angles of a right angled triangle are complements of each other; whence, C = 90° − B = 46° 35′. *To find* AB, we have $\cos B = \frac{AB}{BC}$, or AB = 235 × cos 43° 25′. Now, from the table of natural functions we find cos 43° 25′ = .72637; whence AB = 235 × .72637 = 170.7. *To find* AC, we have $\sin B = \frac{AC}{BC}$; whence AC = 235 × .6873 = 161.52.

2. In a right angled triangle ABC, the hypotenuse AC is 94.6, and the angle C is 56° 30′. Find the angle A, and the sides AB and BC.

A = 33° 30′; AB = 78.88; BC = 52.21.

3. In a right angled triangle BDF, the hypotenuse BF is 127.9, and the angle B is 40° 10′ 30″. Find the angle F, and the sides BD and DF.

F = 49° 49′ 30″; BD = 97.72; DF = 82.51.

4. In the triangle CDE, right angled at E, given the side DE 75, the side CE 50.59, to find the other parts.

Hypotenuse = 90.47.

5. In the right angled triangle CDE, given the hypotenuse CD 264, the side CE 135.97, to find the other parts.

DE = 226.28.

6. Given the hypotenuse 435, and one of the acute angles 44°, to find the other parts.

7. Given the hypotenuse 64, and the base 51.778, to find the other parts.

8. Given the hypotenuse 749 feet, and the base 548.255 feet, to find the other parts.

9. Given the hypotenuse 125.7 yards, and one of the acute angles 75° 12′ 23″, to find the other parts.

10. Given one side 388.875, and the adjacent angle 27° 38′ 50″, to find the other parts of a right angled triangle.

(*b*) BY MEANS OF A TABLE OF LOGARITHMIC FUNCTIONS.

11. In a right angled triangle, given an oblique angle 54° 27′ 39″, and the side opposite 56.293, to find the other parts.

SOLUTION.—The other oblique angle is $90° - 54° 27' 39'' = 35° 32' 21''$.

To find the hypotenuse, we have $\sin 54° 27' 39'' = \frac{56.293}{hy}$, or $hy = \frac{56.293}{\sin 54° 27' 39''}$. Applying logarithms to facilitate computation, $\log hy = \log 56.293 - \log \sin 54° 27' 39'' + 10$. The 10 is added since the log sin 54° 27′ 39″ found in the table is 10 too great. Now, looking in Table I., we find $\log 56.293 = 1.750454$; and in Table II., $\log \sin 54° 27' 39'' = 9.910473$. Hence,

$$\log hy = 1.750454 - 9.910474 + 10 = 1.839980, \text{ and } hy = 69.18.$$

To find the other side we have, $\tan \text{angle} = \frac{side\ opp}{side\ adj}$, or $\tan 54° 27' 30'' = \frac{56.293}{side\ adj}$; whence $side\ adj = \frac{56.293}{\tan 54° 27' 30''}$. Applying logarithms, $\log side\ adj = \log 56.293 - \log \tan 54° 27' 39'' + 10 = 1.750454 - 10.146104 + 10 = 1.604350$.
$\therefore$ *Side adj* $= 40.2115$.

12. In a right angled triangle ABC, the hypotenuse AC is 340, and the side AB is 200. Find the acute angles A and C, and the perpendicular BC.

A = 53° 58′ 6″; C = 36° 1′ 54″; BC = 274.95.

13. In a right angled triangle ABC, the perpendicular AB is 736.3 and BC 500. Find the acute angles A and C, and the hypotenuse AC.

A = 34° 10′ 45″; C = 55° 49′ 15″; AC = 890.02.

14. In a right angled triangle BDF, the perpendicular BD is 246.32, and DF 380.07. Find the acute angles E and F, and the hypotenuse BF.

B = 57° 3′ 11″; F = 32° 56′ 49″; BF = 452.91.

15. In a right angled triangle ABC, the side AB is 249, and the angle A is 29° 14′. Find the perpendicular BC, and the hypotenuse AC.

BC = 139.35; AC = 285.341.

16. In a right angled triangle ABC, the hypotenuse AC is 95.75, and the side BC 60. Find the acute angles A and C, and the perpendicular AB.

$A = 38° 48' 7''$; $C = 51° 11' 53''$; $AB = 74.62$.

17. In a right angled triangle ABC, the side BC is 364.3, and the angle A is 50° 45′. Find the perpendicular AB, and the hypotenuse AC.

$AB = 297.645$; $AC = 470.433$

[NOTE.—The first ten examples may be solved by logarithms if additional exercises are needed, or these by means of the natural functions. Also any one of the examples will afford several others by giving and requiring different parts. Thus, from *Ex.* 17, we can give AB = 297.645, A = 50° 45′, and require the other parts, etc.].

GENERAL APPLICATIONS.

1. Find the area of a parallelogram whose adjacent sides are 28 and 30 feet, and the included angle 75°.

SUG.—First find the altitude.

2. A railroad track 463 feet 3 inches in length has a uniform grade of 3°. Show that the vertical rise is 24 feet 3 inches, nearly.

3. A railroad track makes a vertical rise of 150 feet, by uniform grade, in 3,000 feet of track. What is the grade?

4. Find the apothem and radius of the circumscribed circle of a regular heptagon one of whose sides is 12 feet.

5. Find the area of a regular dodecagon inscribed in a circle whose radius is 12.

6. The angle of elevation to the top of a steeple is 47° 30′, as measured from a point in the same horizontal plane as its base, and at a distance of 200 feet from it. What is the height of the steeple?

Ans. 218.26. ft.

7. A tower 103 feet high throws a shadow 51.5 feet long, upon the horizontal plane of its base: what is the angle of elevation of the sun?

8. The angle at the vertex of a right cone is 52° 23′, and the slant height 126 feet: what is the diameter of the base, and what the altitude?

9. In *Fig.* 13, letting EO represent the earth and M the moon, the radius of the earth $EO = 3956.2$, and the angle EMO * $= 57'$, required to find the distance OM, E being a right angle.

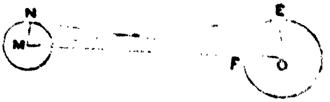


FIG. 13.

The distance of the moon from the earth, as given by this computation, is 238,613 miles.

10. In *Fig.* 13, letting ON represent a tangent to the moon's disc at N, the angle NOM is readily measured, being half the moon's apparent diameter. The apparent diameter of the moon being 31′ 20″, and its distance from the earth as found in the last example, what is the diameter?

Ans. 2176 miles.

OF OBLIQUE ANGLED PLANE TRIANGLES.

IMPORTANT RELATIONS EXISTING BETWEEN THE SIDES AND TRIGONOMETRICAL FUNCTIONS OF THE ANGLES OF OBLIQUE ANGLED PLANE TRIANGLES.

85. Prop.—*The sides of any plane triangle are proportional to the sines of the angles opposite.*

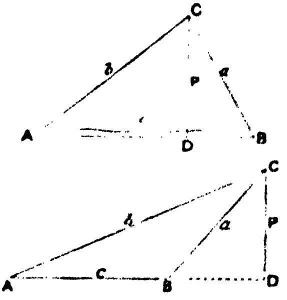


FIG. 14.

DEM.—Let ABC be any plane triangle. Let fall from either angle, as C, a perpendicular upon the opposite side, or upon that side produced. Designate the angles by A, B, and C, the sides opposite by a, b, and c, and the perpendicular by P.

Now, from the right angled triangle ADC we have $P = b \sin A$; also from CDB, $P = a \sin B$; sin ABC in the second figure being = sin CBD. Hence, equating the values of P, $b \sin A = a \sin B$, or $a : b :: \sin A : \sin B$. Q. E. D.

---

* This angle is called the moon's horizontal parallax, and is readily measured. Some rude notion of the manner in which parallax becomes apparent, may be got from considering the difference in direction from two observers to the moon, one observer standing directly under the moon, as at F, and the other at E, seeing the moon in his horizon. The angular displacement of the moon due to these different points of observation is horizontal parallax.

***86. Prop.**—The sum of any two sides of a plane triangle is to their difference, as the tangent of half the sum of the angles opposite is to the tangent of half their difference.*

Dem.—Letting a and b represent any two sides of a plane triangle, and A and B the angles opposite, we have $a : b :: \sin A : \sin B$. Taking this both by composition and division, we have $a + b : a - b :: \sin A + \sin B : \sin A - \sin B$. But from (*60*), $\sin A + \sin B : \sin A - \sin B :: \tan\frac{1}{2}(A + B) : \tan\frac{1}{2}(A - B)$. $\therefore\ a + b : a - b :: \tan\frac{1}{2}(A + B) : \tan\frac{1}{2}(A - B)$. Q. E. D.

***87. Prop.**—The tangent of half of any angle of a plane triangle equals k, divided by half the perimeter of the triangle minus the side opposite the angle; in which k is the radius of the inscribed circle, and equals the square root of the continued product of half the perimeter minus each side separately, divided by half the perimeter.*

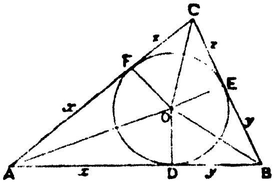


Fig. 15.

Dem.—Let ABC be any plane triangle. Represent the angles by A, B, and C, the sides opposite by a, b, and c, the perimeter by p, and the segments of the sides made by the radii of the inscribed circle, by x, y, and z, as in the figure.

Then $a + b + c = 2x + 2y + 2z = p$, or $x + y + z = \frac{1}{2}p$.

Whence $x = \frac{1}{2}p - a$, $y = \frac{1}{2}p - b$, and $z = \frac{1}{2}p - c$; since $y + z = a$, $x + z = b$, and $x + y = c$.

Now from the triangles AOD, DOB, and COE, $\tan\frac{1}{2}A = \frac{k}{x} = \frac{k}{\frac{1}{2}p - a}$, $\tan\frac{1}{2}B = \frac{k}{y} = \frac{k}{\frac{1}{2}p - b}$, and $\tan\frac{1}{2}C = \frac{k}{z} = \frac{k}{\frac{1}{2}p - c}$.

To find k. $\frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C = 90°$, or $\frac{1}{2}A = 90° - (\frac{1}{2}B + \frac{1}{2}C)$; whence, $\tan\frac{1}{2}A = \frac{k}{x} = \tan[90° - (\frac{1}{2}B + \frac{1}{2}C)] = \cot(\frac{1}{2}B + \frac{1}{2}C) = \frac{1 - \tan\frac{1}{2}B\tan\frac{1}{2}C}{\tan\frac{1}{2}B + \tan\frac{1}{2}C}$ (*54*, Dem).

Substituting for $\tan\frac{1}{2}B$, and $\tan\frac{1}{2}C$, $\frac{k}{y}$, and $\frac{k}{z}$, we have $\frac{k}{x} = \frac{1 - \frac{k^2}{yz}}{\frac{k}{y} + \frac{k}{z}}$; whence $k^2 = \frac{xyz - xk^2}{z + y}$; $(x + y + z)k^2 = xyz$; and $k = \sqrt{\frac{xyz}{x + y + z}}$. In this value of k, substituting for x, y, and z their values, we have $k = \sqrt{\frac{(\frac{1}{2}p - a)(\frac{1}{2}p - b)(\frac{1}{2}p - c)}{\frac{1}{2}p}}$.

88. Sch. 1.—These three propositions (***85***, ***86***, ***87***), furnish the most elegant and expeditious means for finding the unknown parts of an oblique angled plane triangle, when a sufficient number of parts are given or known (***83***)

89. Sch. 2.—***Important Practical Suggestions.***

1st. Two angles of a triangle being given, the third is known by implication, it being the supplement of the sum of the other two.

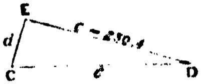
2nd. When two of the known, or given, parts are opposite each other, the first proposition (***85***) effects the solution.

3rd. When two sides and the included angle are given, the solution is effected by means of the second proposition (***86***).

4th. When the three sides are given, the angles are found by the third proposition (***87***).

EXERCISES.

1. In the plane triangle CDE, given the angle D = 15° 19′ 51″, C = 72° 44′ 05″, and the side *c*, opposite C, 250.4, to find the other parts.



Solution.—*First*, E = 180° − (D + C) = 91° 56′ 04″ (***89***, 1st).

Second, To find side *d* opposite angle D (***89***, 2nd).

sin C : sin D : : *c* : *d*, or

sin 72° 44′ 05″ : sin 15° 19′ 51″ : : 250.4 : *d*.

This proposition may be solved for *d*, by taking the natural sine of 15° 19′ 51″, multiplying it by 250.4, and dividing the product by the natural sine of 72° 44′ 05″; or, more expeditiously, by logarithms, as follows:

log 250.4 = ... 2.398634
log sin 15° 19′ 51″ = ... 9.422249
log sin 72° 44′ 05″ (ar. comp.)[*] = ... 0.020024
log *d* = ... [†] 1.840907 ∴ *d* = 69.328.

Third, To find side *e* opposite angle E (***89***, 2nd).

sin C : sin E : : *c* : *e*, or

sin 72° 44′ 05″ : sin 91° 56′ 04″ : : 250.4 : *e*.

Making the computation by logarithms,

log 250.04 = ... 2.398634
log sin 91° 56′ 04″ [‡] 9.999752
log sin 72° 44′ 05″ (ar. comp.) = ... 0.020024
log *e* = ... 2.418410. ∴ *e* = 262.006.

---

[*] See Introduction to Table I. (17).

[†] The student must bear in mind the fact that all the log. trig. funcs. are 10 *too large*, and must see exactly what corrections to make in his results, on this account. In this case the *ar. comp.* is 10 *too small*, since the logarithm we took from the table for log sin 72° 44′ 05″ was 10 too large. But our log sin 15° 19′ 51″ is 10 *too large*, and just corrects the latter. Hence, we have to reject only 10 from the entire sum 11.840907, and this on account of the use of *ar. comp.*

[‡] Take the sine of the supplement, or the *cosine* of the given angle minus 90°.

2. In the plane triangle ABC, $A = 35^\circ\ 42'$, $B = 76^\circ\ 27'$, and $AB = 142$. What are the other parts?

Ans. $C = 67^\circ\ 51'$; $AC = 149.05$; $BC = 89.47$.

3. Given two angles of a plane triangle, $23^\circ\ 40'\ 32''$ and $69^\circ\ 39'\ 51''$, and the included side 100, to find the other parts.

4. In a plane triangle ABC, the side AB is 254.3, the side AC 396.8, and the angle B $94^\circ\ 29'$. Find the angles A and C, and the side BC.

$A = 45^\circ\ 48'\ 21''$; $C = 39^\circ\ 42'\ 39''$; $BC = 285.37$.

SUG'S.—To find the angle C, we have

$$396.8 : 254.3 :: \sin 94^\circ\ 29' : \sin C.$$

From this proportion we get $\log \sin C = 9.805443$. Now, as we have seen before, there are an infinite number of angles corresponding to any given sine, how shall we know what one to take in this case? First, no angle of a triangle can exceed 180°; hence, there are but *two* angles, one an acute angle, and the other its supplement, which can come into consideration in the solution of plane triangles. But which of *these two* are we to take? Thus, in this case, both the angles $39^\circ\ 42'\ 39''$ and $140^\circ\ 17'\ 21''$ correspond to $\log \sin 9.805443$. In this example the ambiguity is resolved by observing that the given angle B is obtuse, and a plane triangle can have but one obtuse angle. $\therefore C = 39^\circ\ 42'\ 39''$.

5. In a plane triangle BDF, the angle B is 40°, the side BD is 400, and the side DF 350. Find the angles D and F, and the side BF.

SUG'S.—To find F, we have

$$350 : 400 :: \sin 40^\circ : \sin F,$$

from which $\log \sin F = 9.866059$, and $F = 47^\circ\ 16'\ 28''$, and its supplement $132^\circ\ 43'\ 32''$. How are we to determine which of these to take? The *given* angle is 40°; hence, as far as that is concerned, either of the two will meet the conditions. There are, therefore, *two* angles, $F = 47^\circ\ 16'\ 28''$ and $F = 132^\circ\ 43'\ 32''$, which fulfill the conditions. We therefore solve two triangles, one having two of its sides 400 and 350, and the angles 40°, $47^\circ\ 16'\ 28''$, and $92^\circ\ 43'\ 32''$; and another triangle with the same given parts and the two required angles $132^\circ\ 43'\ 32''$, and $7^\circ\ 16'\ 28''$. This is readily illustrated geometrically. Thus, lay off $DBF' = 40^\circ$. Take $BD = 400$. Then from D as a centre, with radius 350 describe an arc cutting BF′. It is evident that if B is an acute angle the following cases may arise depending upon the value of DF:

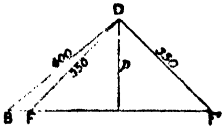


FIG. 16.

1st. If DF is less than the perpendicular p, the problem is impossible.

2nd. If $DF = p$, the triangle is right angled at F.

3rd. If $DF > p$ and $< BD$ there are two triangles, one with an acute angle at F′, as DF′B, and the other with an obtuse angle at F, as DFB, both of which fulfil the conditions of the problem.

4th. If $DF > DB$ there is but one triangle which fulfills the conditions, viz., the one with an acute angle at F′.

The results in the above examples are, for triangle DF'B, angle DF'B = 47° 16′ 28″, BDF′ = 92° 43′ 32″, and side BF′ = 543.89; for triangle DFB, angle DFB = 132° 43′ 32″, angle BDF = 7° 16′ 28″, and side BF = 68.94.

90. COR.—*In applying trigonometrical formulæ to the solution of triangles, if the part sought is found in terms of its* SINE, *the result is ambiguous, and we are to determine whether there really are two solutions to the problem in a geometrical sense, by certain geometrical considerations, or else by trying both values for the angle determined by its sine. This ambiguity arises only when an angle is determined by its sine, as will appear hereafter.*

6. Given two sides of a plane triangle 201 and 140, and the angle opposite the latter 36° 44′. Find the other parts.

Results.—There are two triangles.

Parts of the first, 120° 49′ 49″, 22° 26′ 11″, and 89.34;
Parts of the second, 59° 10′ 11″, 84° 5′ 49″, and 232.84.

7. Given two sides of a plane triangle 180, 100, and the angle opposite the former 127° 33′, to find the other parts.

There is but one triangle, and the parts are 26° 7′ 59″, 26° 19′ 1″, and 100.65.

8. Given two sides of a plane triangle 30.8 and 54.12, and the angle opposite the latter 36° 42′ 11″, to find the other parts. Why but one triangle?

9. Given two sides of a plane triangle 600 and 250, and the angle opposite the latter 42° 12′. Find the other parts.

SUG.—Attempting to get the angle opposite 600, we find log sin = 10.207400, which is impossible. It is in some such way that a trigonometrical solution shows a geometrical absurdity.

10. Given two sides of a plane triangle 1337.5 and 493, and the angle opposite the former 69° 46′. Find the other parts.

---

11. In a plane triangle, given two sides 1686 and 960, and the included angle 128° 04′, to find the other parts.

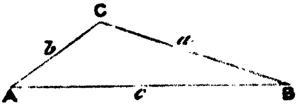


FIG. 17.

SOLUTION.—Let $a = 1686$, $b = 960$, and $C = 128° 04'$. (See **89**, 3rd.) The sum of the angles A and B is $180° - 128° 04' = 51° 56'$, and $\frac{1}{2}(A + B) = 25° 38'$. From (**86**) we have,

$$a + b : a - b :: \tan \tfrac{1}{2}(A + B) : \tan \tfrac{1}{2}(A - B), \text{ or}$$

$$2646 : 726 :: \tan 25° 58' : \tan \tfrac{1}{2}(A - B).$$

Making the computations by logarithms, we find $\log \tan \frac{1}{2}(A - B) = 9.125887$. Hence, $\frac{1}{2}(A - B) = 7° 36' 40''$, the angle found in the table, or its supplement. But $\frac{1}{2}$ the difference of two angles of a triangle is less than 90°; consequently $\frac{1}{2}(A - B) = 7° 36' 40''$.

Now having $\frac{1}{2}(A + B) = 25° 58'$, and $\frac{1}{2}(A - B) = 7° 36' 40''$, we find $A = 33° 34' 40''$, and $B = 18° 21' 20''$.

The side c can be found by (*85*) as two opposite parts are now known. $c = 2400$.

12. In a plane triangle ABC, the side AB is 304, BC 280.3, and the included angle B is 100°. Find the angles A and C, and the side AC.

$A = 38° 3' 3''$; $C = 41° 56' 57''$; $AC = 447.856$.

13. In a plane triangle ABC, the side AB is 103, AC 126, and the included angle A is 56° 30′. Find the angles B and C, and the side BC.

$B = 72° 20' 15''$; $C = 51° 9' 45''$; $BC = 110.267$.

---

14. In a plane triangle ABC, given the three sides, $a = 3459$, $b = 4209$, and $c = 6030.4$, to find the angles.

SOLUTION.—Applying (*87*), we have,

$$k = \sqrt{\frac{(\frac{1}{2}p - a)(\frac{1}{2}p - b)(\frac{1}{2}p - c)}{\frac{1}{2}p}}, \text{ or}$$

$$\log k = \tfrac{1}{2}\{\log(\tfrac{1}{2}p - a) + \log(\tfrac{1}{2}p - b) + \log(\tfrac{1}{2}p - c) - \log \tfrac{1}{2}p\}.$$

Also, $\tan \frac{1}{2}A = \frac{k}{\frac{1}{2}p - a}$, $\tan \frac{1}{2}B = \frac{k}{\frac{1}{2}p - b}$, and $\tan \frac{1}{2}C = \frac{k}{\frac{1}{2}p - c}$, or $\log \tan \frac{1}{2}A = \log k - \log(\frac{1}{2}p - a)$, $\log \tan \frac{1}{2}B = \log k - \log(\frac{1}{2}p - b)$, and $\log \tan \frac{1}{2}C = \log k - \log(\frac{1}{2}p - c)$.

COMPUTATION.

$$\begin{array}{rll}
a = & 3459 & \\
b = & 4209 & \\
c = & 6030.4 & \\
\hline
p = & 13698.4 & \\
\frac{1}{2}p = & 6849.2 & \text{(ar. comp.) } \log = 6.164360 \\
\frac{1}{2}p - a = & 3390.2 & \dots\dots \log = 3.530236 \\
\frac{1}{2}p - b = & 2640.2 & \dots\dots \log = 3.421637 \\
\frac{1}{2}p - c = & 818.8 & \dots\dots \log = 2.913178 \\
 & & \qquad 2)\overline{6.029401} \\
 & & \log k = 3.014700
\end{array}$$

$\log \tan \frac{1}{2}A = \log k - \log(\frac{1}{2}p - a) = 9.484474 \quad \therefore A = 33° 56' 10''.5$

$\log \tan \frac{1}{2}B = \log k - \log(\frac{1}{2}p - b) = 9.593063 \quad \therefore B = 42° 47' 25''.3$

$\log \tan \frac{1}{2}C = \log k - \log(\frac{1}{2}p - c) = 10.101523 \quad \therefore C = 103° 16' 24''.2$

Proof, $A + B + C = 180° 00' 00''$

15. In a plane triangle **ABC**, the side **AB** is 95.6, **BC** is 275, and **AC** 300. Find the angles **A**, **B**, and **C**.

$A = 65° 47' 55''$; $B = 95° 42' 52''$; $C = 18° 29' 13''$.

16. In a plane triangle **BDF**, the side **BD** is 500, **DF** is 403.7, and **BF** 395.75. Find the angles **B**, **D**, and **F**.

$B = 52° 0' 3''$; $D = 50° 34' 45''$; $F = 77° 25' 12''$.

OBLIQUE ANGLED TRIANGLES SOLVED BY MEANS OF RIGHT ANGLED TRIANGLES.

[NOTE.—Articles *91-94* inclusive, may be omitted in an elementary course, if thought desirable. Or *91* and its applications may be taken instead of *86-88*. *85* should be included in any course. It is too elementary and important to be omitted.]

91. Prop.—*All cases of oblique angled plane triangles may be solved by the solution of right angled triangles.*

DEM.—Of the three given parts we may affirm that they are, 1st, All adjacent; 2nd, Two adjacent and one separated; or 3rd, All separated.

1st. *When the given parts are all adjacent*; *i. e.*, when they are *two sides* and the *included angle*, or *two angles* and the *included side*. To solve the first let fall a perpendicular from the extremity of one of the given sides upon the other given side, or upon that side produced. There will thus be formed two right angled triangles which can be computed, and from the parts of which the parts of the required triangle can be found. Thus, let **A** be the given angle, and b and c the given sides. In the right angled triangle **ADC** there are given **A** and b, whence **AD**, **P**, and angle **ACD**, can be computed. Then passing to triangle **CDB**, we know **P**, and **DB** (since we have c given and have computed **AD**). Hence, we can compute **B**, a, and **DCB**. Thus, the parts of **ACB** become known . . . . When the given parts are two angles and the included side, find the third angle by taking the supplement of the two given. Let fall a perpendicular from one extremity of the given side upon the opposite side. The two right angled triangles thus formed can then be computed. Thus, if **A**, b, and **C** are given, having found **B**, let fall **CD**. The triangle **ACD** has the angle **A** and side b known, whence its parts can be computed. Having computed **P** we can pass to the triangle **CDB**, and knowing **P** and **B**, can compute it. Thus the parts of **ACB** become known.

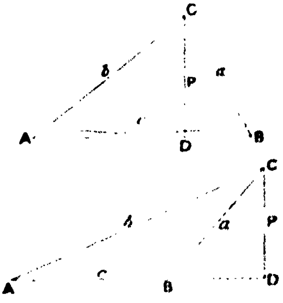


FIG. 18.

2nd. *When two of the given parts are adjacent and one separated; i. e*, when two angles and a side opposite one are given; or two sides and an angle opposite one are given. The first of these cases is virtually the same as the last given. To solve the other, let fall a perpendicular from the angle between the given sides, and two right angled triangles will be formed which can readily be computed. Thus a, b, and A being given, and CD let fall from C, the triangle ACD can first be computed, and then CDB. This is the ambiguous case, but it is easily determined. Having computed P, if the given side a is less than P there is *no* solution; if = to P, *one* solution (a right angled triangle); if $a >$ P and $< b$, there are two solutions, *i. e.*, it will go in *between* CD and AC, and also beyond CD; if $a >$ P and also $> b$ there is only *one* solution, as it will not go in between CD and AC.

3rd. *When the three given parts are all separated from each other.* This is the case in which the three sides are given to find the angles. It is readily solved by letting fall a perpendicular from the angle opposite the greatest side, upon that side, as CD upon AB. Then compute the segments AD (which call m), and DB (n), from the following relation (PART II, Ex. 12, page 162):

$$m + n \text{ (or } c) : b + a :: b - a : m - n.$$

Knowing m and n, the angles of the two right angled triangles ACD and CDB can be computed, and these make known the angles of ACB. Q. E. D.

[NOTE.—A few additional examples are here given which the pupil can use to illustrate the theory presented in (*91*). If more are needed the preceding can be used: these may also be used to apply the methods before given. Again, a very great variety and number of examples may be made from these by assigning different parts as known.]

EXERCISES.

1. In a plane triangle BDF, the side BF is 123.75, DF 500, and the included angle F 120°. Find the angles B and D, and the side BD.

B = $49^\circ\ 12'\ 4''$; D = $10^\circ\ 47'\ 56''$; BD = 572.006.

2. In a plane triangle ABC, the angle A is $70^\circ\ 21'$, the angle B $54^\circ\ 22'$, and the side BC 125. Find the angle C, and the sides AB and AC.

C = $55^\circ\ 17'$; AB = 109.1; AC = 107.88.

3. In a plane triangle ABC, the side AB is 98, the side BC 95.12, and the angle C $33^\circ\ 21'$. Find the angles A and B, and the side AC.

A = $32^\circ\ 14'\ 55''$; B = $114^\circ\ 24'\ 5''$; AC = 162.33.

4. In a plane triangle DAC, given AD = 450, AC = 309, and D = $27^\circ\ 50'$, to find the other parts.

C = $137^\circ\ 9'\ 36''$, or $42^\circ\ 50'\ 24''$; A = $15^\circ\ 0'\ 24''$, or $109^\circ\ 19'\ 36''$; DC = 171.36, or 624.5.

COMPUTATION.

It will give definiteness to the student's thought, if he first sketch the figure geometrically. Thus, lay off D = 27° 50′, and taking AD = 450, let fall the perpendicular AP.

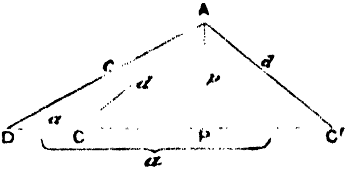


FIG. 19.

1st. *To compute p.*

$p = c \sin D = 450 \sin 27° 50′$,

$\log 450 = 2.653213$

$\log \sin 27° 50′ = 9.669225$

$\log p = 2.322438$. $\therefore p = 210.106$.

Knowing p, we see by inspection that AC can lie in both the positions AC and AC′, and hence that there are two solutions.

2nd. *To compute* C, from the triangle ACP, in which d and p are now known.

$$\text{Sin } C = \frac{p}{d} = \frac{210.106}{309}$$

$\log 210.106 = 2.322438$

$\log 309 = 2.489958$

$\log \sin C = 9.832480$. $\therefore C′ = 42° 50′ 24″$, and $C = 137° 9′ 36″$.

3rd. *To find the angle* A. DAC = 180° − (D + ACD) = 180° − 164° 59′ 36″ = 15° 0′ 24″. DAC′ = 180° − (D + AC′D) = 180° − 70° 40′ 24″ = 109° 19′ 36″.

4th. *To find* DC. Compute DP and CP from the triangles APD and APC. DP − CP = DC, and DP + CP = DC′.

5. In a plane triangle ABC, the side AB is 460, BC is 340, and AC 280. Find the angles A, B, and C.

A = 47° 23′ 16″; B = 37° 18′ 31″; C = 95° 18′ 13″.

6. The sides of a plane triangle are 40, 34, and 25 feet respectively; required the angles.

38° 25′ 20″, 57° 41′ 24″, 83 53′ 16″.

7. The sides of a plane triangle are 390, 350, and 270 feet respectively; required the angles.

42° 22′ 06″, 60° 52′ 33″, and 76° 45′ 21″.

8. Given two sides of a plane triangle 450 and 540, and the included angle 80°, to find the remaining parts.

Angles, 56° 11′, 43° 49′; and the side, 640.08.

9. Given two sides of a plane triangle 76 and 109, and the included angle 101° 30′, to find the remaining parts.

Angles, 30° 57′ 30″, 47° 32′ 30″; and the side, 144.8.

FUNCTIONS OF THE ANGLES OF A TRIANGLE IN TERMS OF THE SIDES.

92. Prop.*—Any side of a plane triangle equals the sum of the products of each of the other sides into the cosine of the angle which it makes with the first side.*

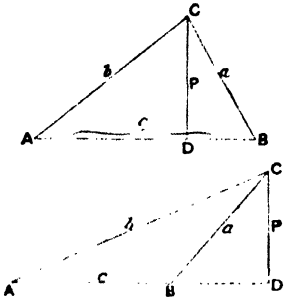


FIG. 20.

DEM.—In the first figure $AB = AD + DB$. But $AD = b\cos A$, and $DB = a\cos B$. $\therefore c = b\cos A + a\cos B$. In the second figure $AB = AD - DB$. But $AD = b\cos A$, and $DB = a\cos CBD = a(-\cos CBA) = -a\cos B$. $\therefore c = b\cos A - (-a\cos B) = b\cos A + a\cos B$. In like manner, we have $a = b\cos C + c\cos B$, and $b = a\cos C + c\cos A$. Collecting and arranging,

$$(1)\ a = b\cos C + c\cos B;$$
$$(2)\ b = a\cos C + c\cos A;$$
$$(3)\ c = a\cos B + b\cos A. \quad \text{Q. E. D.}$$

93. COR.—*The square of any side of a plane triangle equals the sum of the squares of the other two, minus twice their rectangle into the cosine of their included angle.*

DEM.—From (3) (*92*), we have by transposing and squaring,

$$a^2\cos^2 B = c^2 + b^2\cos^2 A - 2bc\cos A; \text{ and}$$

from (*85*) $a^2\sin^2 B = \qquad b^2\sin^2 A.$

$$\text{Adding,}\quad a^2 = c^2 + b^2 - 2bc\cos A.$$
$$\text{In like manner,}\ b^2 = a^2 + c^2 - 2ac\cos B;$$
$$\text{and}\quad c^2 = a^2 + b^2 - 2ab\cos C. \quad \text{Q. E. D.}$$

94. SCH.—These *formulæ* afford another means for finding the angles of a plane triangle when the sides are given. Thus,

$$(1_1)\ \cos A = \frac{b^2 + c^2 - a^2}{2bc};$$

$$(2_1)\ \cos B = \frac{a^2 + c^2 - b^2}{2ac};$$

$$(3_1)\ \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

These *formulæ* give directly the natural cosines of the angles in terms of the sides. To adapt them to logarithmic computation, we transform them as follows:

Subtracting each member of (1_1) from unity, $1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} =$

$$\frac{a^2 - b^2 - c^2 + 2bc}{2bc} = \frac{a^2 - (b - c)^2}{2bc} = \frac{\{a + (b - c)\}\{a - (b - c)\}}{2bc} = \frac{(a + b - c)(a + c - b)}{2bc}$$

But $1 - \cos A = 2\sin^2 \frac{1}{2}A$ (57, O); and letting $p = a + b + c$, $\frac{1}{2}(a + b - c) = \frac{1}{2}p - c$, and $\frac{1}{2}(a + c - b) = \frac{1}{2}p - b$. Whence, substituting, $2\sin^2 \frac{1}{2}A = \dfrac{4(\frac{1}{2}p - c)(\frac{1}{2}p - b)}{2bc}$, or

$$(1_2)\ \sin \tfrac{1}{2}A = \sqrt{\frac{(\frac{1}{2}p - c)(\frac{1}{2}p - b)}{bc}}.$$ In like manner,

$$(2_2)\ \sin \tfrac{1}{2}B = \sqrt{\frac{(\frac{1}{2}p - c)(\frac{1}{2}p - a)}{ac}};$$ and

$$(3_2)\ \sin \tfrac{1}{2}C = \sqrt{\frac{(\frac{1}{2}p - b)(\frac{1}{2}p - a)}{ab}}$$

In a manner altogether similar, by adding each member of (1_1) to unity, and reducing, we get

$$(1_3)\ \cos \tfrac{1}{2}A = \sqrt{\frac{\frac{1}{2}p(\frac{1}{2}p - a)}{bc}};$$ and from (2_1) and (3_1),

$$(2_3)\ \cos \tfrac{1}{2}B = \sqrt{\frac{\frac{1}{2}p(\frac{1}{2}p - b)}{ac}};$$

$$(3_3)\ \cos \tfrac{1}{2}C = \sqrt{\frac{\frac{1}{2}p(\frac{1}{2}p - c)}{ab}}.$$

Dividing (1_2) by (1_3), (2_2) by (2_3), and (3_2) by (3_3), we have.

$$(1_4)\ \tan \tfrac{1}{2}A = \sqrt{\frac{(\frac{1}{2}p - c)(\frac{1}{2}p - b)}{\frac{1}{2}p(\frac{1}{2}p - a)}};$$

$$(2_4)\ \tan \tfrac{1}{2}B = \sqrt{\frac{(\frac{1}{2}p - c)(\frac{1}{2}p - a)}{\frac{1}{2}p(\frac{1}{2}p - b)}};$$

$$(3_4)\ \tan \tfrac{1}{2}C = \sqrt{\frac{(\frac{1}{2}p - b)(\frac{1}{2}p - a)}{\frac{1}{2}p(\frac{1}{2}p - c)}}.$$

EXERCISES.

[NOTE.—In order to render these *formulæ* familiar, and to give the student exercise in applying *formulæ*, a few examples are appended. If necessary, any which precede can be used.]

1. The sides of a plane triangle being 40, 34, and 25, find the angles.

SOLUTION.—*By natural functions.*

Let the sides be represented by a, b, and c in order, and the angles opposite by A, B, and C; then

$$\operatorname{Cos} A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{1156 + 625 - 1600}{1700} = .10647.\quad \therefore A = 83°\ 53'\ 10''.$$

There is no ambiguity in this case, since the cosine is +, and hence the angle is < 90°.

The same angle is found by logarithmic computation, thus:

$a = 40$ . . . log = 1.602060	$\log(\tfrac{1}{2}p - c) = 1.389166$
$b = 34$ . . . log = 1.531479	$\log(\tfrac{1}{2}p - b) = 1.190332$
$c = 25$ . . . log = 1.397940	a. c. $\log b = 8.468521$
$p = 99$ . . . log = 1.995635	a. c. $\log c = 8.602060$
$\tfrac{1}{2}p = 49.5$ . . . log = 1.694605	$2)\bar{1}.650079$
$\tfrac{1}{2}p - a = 9.5$ . . . log = 0.977724	$\bar{1}.825039$
$\tfrac{1}{2}p - b = 15.5$ . . . log = 1.190332	
$\tfrac{1}{2}p - c = 24.5$ . . . log = 1.389166	$\log \sin \tfrac{1}{2}A = 9.825039.$

$\therefore\ \tfrac{1}{2}A = 41°\ 56'\ 38''$ and $A = 83°\ 53'\ 16''$.

In like manner the other angles may be found.

2. The sides of a plane triangle being 6, 5, and 4, find the angles.

The angles are 82° 49′ 09″, 55° 46′ 16″, and 41° 24′ 35″.

3. The sides of a plane triangle being 8601.5, 4082, and 7068, find the angles.

The angles are 54° 35′ 12″, 28° 4′ 44″, and 97° 20′ 4″.

4. The sides of a plane triangle being .5123864, .3538971, and .3090507, find the angles.

The angle opposite the last side is 36° 18′ 10″.2.

AREA OF PLANE TRIANGLES.

95. Prop.—*The area of a plane triangle is equal to half the product of any two sides into the sine of the included angle.*

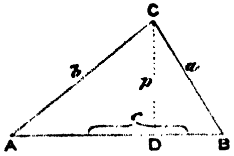


FIG. 21.

DEM.—Let ABC be any plane triangle, and b and c any two sides with A as the included angle. From the extremity of one of these two sides remote from A, let fall a perpendicular p, upon the other side Now,

$$\text{Area ACB} = \tfrac{1}{2}pc.$$

But, from ACD, $p = b \sin A$. ∴ Area ACB = $\tfrac{1}{2}bc \sin A$. Q. E. D.

96. COR.—*The area of a plane triangle is equal to the square root of the continued product of half its perimeter into half its perimeter minus each side separately.*

Dem.—From the proposition, and since $\sin A = 2\sin\frac{1}{2}A\cos\frac{1}{2}A$, we have,

$$\text{Area} = \tfrac{1}{2}bc\sin A = bc\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A = bc\sqrt{\frac{(\frac{1}{2}p-c)(\frac{1}{2}p-b)}{bc}} \times \sqrt{\frac{\frac{1}{2}p(\frac{1}{2}p-a)}{bc}} =$$

$$\sqrt{\tfrac{1}{2}p(\tfrac{1}{2}p-a)(\tfrac{1}{2}p-b)(\tfrac{1}{2}p-c)}.$$ Q. E. D.

EXERCISES.

1. Given two sides of a plane triangle 125.81 and 57.65, and the included angle 57° 25′. Find the area.

Area = 3055.7.

2. Given the sides of a plane triangle 103.5 and 90, and the included angle 100°, to find the area.

Area = 4586.74.

3. How many square yards are there in a triangle whose sides are 30, 40, and 50 feet?

Area = 66⅔.

4. Find the area of a triangle whose sides are 20, 30, and 40.

Area = 290.4737.

5. What is the area of a triangle whose sides are 30 and 40, and their included angle 28° 57′?

Area = 290.427.

6. What is the number of square yards in a triangle, of which the sides are 25 feet and 21.25 feet, and their included angle 45°?

Area = 20.8694.

7. Find the area of a triangle in which two of the angles are 80° and 60° respectively, and the included side 32 feet.

Area = 679.33 square feet.

8. Find the area of a triangular field having one of its sides 45 poles in length, and the two adjacent angles, respectively, 70° and 69° 40′.

Area = 1378.411 square poles.

9. Find the area of a triangular piece of ground having two angles respectively 73° 10′ and 90° 50′, and the side opposite the latter 75.3 poles.

Area = 748.03 square poles.

PRACTICAL APPLICATIONS.

[Note.—The following problems are inserted, not as any part of a treatise upon the subject of trigonometry as pure science, but as affording the student good mental exercise, and valuable and interesting information.]

1. To find the length (in miles) of a degree of longitude at Ann Arbor, Mich.

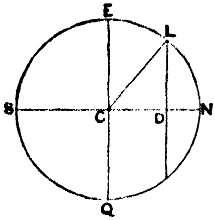


FIG. 22.

SOLUTION.—Let NESQ be a meridian section of the earth, EQ the equatorial diameter, and EL the latitude of Ann Arbor, 42° 16′ 48″.3. A degree of longitude at L is $\frac{1}{360}$ of the circumference of the circle whose radius is LD. CL the radius of the earth at this point = 3957.* Now in the right angled triangle LCD, we have CLD = ECL = 42° 16′ 48″.3, and CL = 3957; whence, LD = CL × cos 42° 16′ 48″.3, and LD = 2927.6. ∴ A degree = 51.1 miles. As a degree in longitude makes 4 minutes difference in time, 51.1 miles east or west on this parallel is equivalent to 4 minutes difference in time.

QUERY.—How does it appear from the above solution that the length of a degree of longitude varies as the cosine of the latitude?

2. To find the distance of a planet from the earth at any particular time.

SOLUTION —To render the problem as simple as possible, we will suppose two observatories on the same meridian, at N, and N′; and that when the planet P is on the same meridian, the angles ZNP, and Z′N′P (the zenith distances) are measured. With these data and the radius of the earth, CN, CN′, known, the problem comes quite within the scope of the present study. The process is as follows: The arc NN′ being known, the angle NCN′ is known. Then in the triangle NCN′, two sides and the included angle are known, whence the other parts can be found.

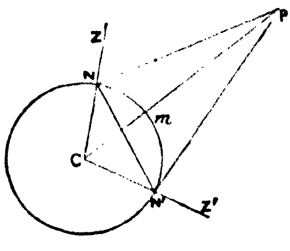


FIG. 23.

Now, knowing the angles PNC, PN′C, and CNN′, CN′N, we can find the angles PNN′, PN′N. This affords sufficient parts of the triangle PNN′ to determine the triangle, and we find PN, or PN′. Finally, in the triangle PNC, we know PN, NC, and the included angle; whence the other parts can be computed. But PC is the distance sought.

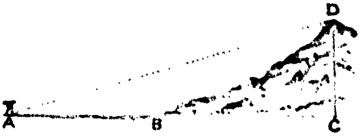
3. Suppose in case of the moon, the angles PNZ, and PN′Z′, being measured, are found to be respectively 44° 54′ 21″, and 48° 42′ 57″, the distance between the points of observation N and N′ is 92° 14′, and the radius of the earth is 3956.2 miles; find the distance to the moon.

Distance = 237,954.098 miles.

---

* The equatorial radius of the earth is 3962.8 miles; but in consequence of the flattening in the direction of the polar diameter it is less here.

4. Required the height of a hill **D** above a horizontal plane **AB**, the distance between **A** and **B** being equal to 975 yards, and the angles of elevation at **A** and **B** being respectively 15° 36′ and 27° 29′.



DC = 587.61 yards

5. Find the area of a regular hexagon, and also of a regular octagon, whose sides are each 10 feet.

Areas, 259.8, and 482.84 square feet

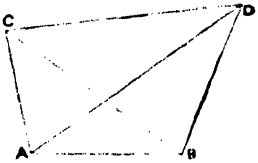
6. Find the area of a regular pentagon, and also of a regular decagon, whose sides are each 12 feet.

Areas, 247.74, and 1107.96 square feet.

7. Wishing to know the length of a certain pond of water, I measured a line 100 yards in length, and at each of its extremities observed the angles subtended by the other extremity and a couple of trees at the extremities of the pond. These angles were, at one end of the line, 32° and 98°, and at the other, 37° and 118°; what was the length of the pond?

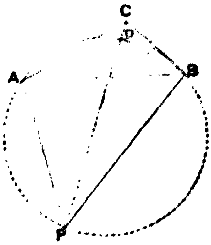
Draw the horizontal line **AB** equal to 100; make the angle **BAD** 32°, **BAC** 98°, **ABC** 37°, and **ABD** 118°. The intersections of the lines **AC** and **BC**, **AD** and **BD**, determine the extremities of the pond; the straight line **CD** is the length of the pond.

CD = 161.868 yards.



8. The distances **AB**, **AC**, and **BC**, between the points **A**, **B**, and **C**, are known; viz., **AB** = 800 yds., **AC** = 600 yds., and **BC** = 400 yds. From a fourth point **P**, the angles **APC** and **BPC** are measured; viz., **APC** = 33° 45′,
and **BPC** = 22° 30′.

Required the distances **AP**, **BP**, and **CP**.



Distances, { **AP** = 710.193 yds.
BP = 934.291 yds.
CP = 1042.522 yds.

SUG'S.—Conceive the circumference passed through A, B, and P, and AD and DB drawn. In the triangle ADB, angle DAB = the given angle DPB, and DBA = APD. Hence, all the parts of triangle ADB can be found. Again, since the sides of the triangle ACB are given, its angles can be found. Then, since angle CAB — DAB = CAD, there are two sides and the included angle known in triangle ACD; whence angle ACD can be found. Thus we reach the triangle ACP, in which there are now known AC and the angles.

9. From the top of a mountain, three miles high, the angle of depression of a line tangent to the earth's surface is taken, and found to be 2° 13′ 27″. What is the diameter of the earth, considered as a sphere?

Ans., 7958.3 miles.

10. Taking the sun's mean apparent diameter as 32′ 3″.4, and his distance from the earth 91,430,000 miles, show that, if his centre were coincident with the earth's, his body would extend in all directions nearly 200,000 miles beyond the moon. (See Ex. 3.)

Sun's diameter = 852,574 miles.

11. Assuming the height of the Great Pyramid to be 486 feet, how far off may it be seen across the desert?

Ans., 27 miles.

12. What was the perpendicular height of a balloon, when its angles of elevation were 35° and 64°, as taken by two observers on the same level, at the same time, both on the same side of it, and in the same vertical plane; the distance between the two observers being 880 yards?

Ans., 935.757 yards.

13. Given two sides of a parallelogram 60 and 80, and a diagonal 100. Is this the longer or shorter diagonal? What is the other? What are the angles of the parallelogram?

14. A balloon being directly over one of two towns standing on the same horizontal plane, at a distance of eight miles from each other, the angle of depression to the more remote town was observed by the æronaut to be 10°. What was the height of the balloon?

Ans., 1.41 miles.

15. The most recent observations make the sun's horizontal parallax 8″.94, and the earth's equatorial radius 3962.8 miles. Show that the distance of the sun from the earth is nearly as given in Ex. 10, instead of 95 millions of miles, as it has been heretofore considered.

CHAPTER II.

SPHERICAL TRIGONOMETRY.

INTRODUCTION.

PROJECTION OF SPHERICAL TRIANGLES.

97. *To Project a Spherical Triangle* on a plane surface is to draw the triangle on that surface so that it will present the same appearance to the eye, situated at a particular point, as when drawn on the surface of a sphere.

98. *The Simplest Method* of projecting a spherical triangle is to project it *on the plane of one of its sides*, the eye being supposed situated in the axis of the sphere perpendicular to this plane, and at an infinite distance from it. The plane is called the *Plane of Projection;* and its intersection with the sphere is called the *Primitive Circle*, and is the base of the hemisphere on which the triangle is conceived as situated.

99. *Fundamental Propositions.*—1st. *When the parts of a spherical triangle are each conceived as less than 180°, any such triangle can be represented on a hemisphere.*

2d. *The primitive Circle has its axis, and consequently its pole, projected at its centre.*

3d. *The semi-circumference of any circle of the sphere, perpendicular to the Primitive Circle, is projected in the chord representing the intersection of the circles; and, if the perpendicular circle be a great circle, its semi-circumference is projected in a diameter of the Primitive Circle.*

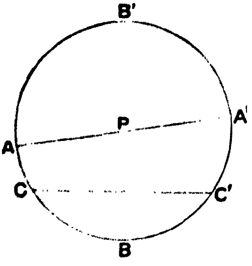


FIG. 24.

ILL'S.—These propositions are direct consequences of the fundamental conception. Thus, let ABA'B' represent the base of the hemisphere on which the triangle is conceived as situated. This is the Primitive Circle, and the eye is supposed situated at an infinite distance, and in a line perpendicular to the plane of the paper at P. The pole of the Primitive Circle being in this line is projected (seen as) at P. As all great circles perpendicular to the Primitive Circle pass through its pole and include its axis, the eye is in all such planes, and any lines of these planes, as the semi-circumferences of the great circles in which they intersect the sphere, are projected (appear to the eye) as diameters of the Primitive Circle. Moreover, since the eye is *at infinity*, it is to be conceived as in the plane of any small circle which is perpendicular to the primitive, and which is therefore projected in a chord, as CC'.

PROJECTION OF RIGHT ANGLED SPHERICAL TRIANGLES.

100. Prob. 1.—*To project a right angled spherical triangle on the plane of one of its sides, when the two sides about the right angle are given.*[*]

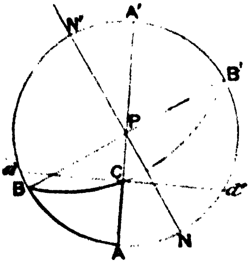


FIG. 25.

SOLUTION.—Let the angles of the triangle be represented by A, B, and C, A being the right angle. Let the sides opposite these angles respectively be represented by a, b, and c; whence b and c are the given sides. Draw the primitive circle and the diameters BB', NN' at right angles to each other. From B lay off BA $= c$.[†] Let the right angle be at A; whence the side b is perpendicular to the primitive circle, and projected in the diameter AA'. To project the vertex C, conceive the semi-circumference, of which AA' is the projection, to revolve on AA' until it falls upon the semi-circumference A'BA, then will the point C fall at d. Hence make $Ad = b$. In like manner revolving the semi-circumference, of which AA' is the projection, until it falls upon A'B'A, the point C will fall at d'. Hence make $Ad' = b$. The point C will describe the semi-circumference of a small circle perpendicular to the primitive circle, and whose projection is dd'. Now, as the projection of the vertex of the triangle **C** is at the same time in **AA′** and *dd′*, it must be at their intersection. Finally, the hypotenuse *a* is projected in a curve passing through **B**, **C**, and **B′**, since two great circles intersect at the extremities of a diameter. This curve is really an ellipse, but for our present purpose it may be considered as the arc of a circle passing through **B**, **C**, and **B′**. Therefore, **BAC** is the projection required.

---

[*] In this treatise the discussions embrace only such triangles as have each part less than 180°.

[†] For the purposes for which we shall use these projections, an arc can be laid off with sufficient accuracy by observing its relation to 90°, 60°, 30°, or some aliquot part of the circumference, which is readily obtained on geometrical principles.

QUERIES.—Will a solution of this problem be possible for all values of *b* and *c*? How does it appear from the projection?

EXAMPLES.

1. Having given $b = 110°$, and $c = 60°$, to project the triangle. See *Fig.* 26.

2. Having given $b = 50°$, and $c = 130°$, to project the triangle.

3. Having given $b = 90°$, and $c = 30°$, to project the triangle.

4. Having given $b = 90°$, and $c = 90°$, to project the triangle.

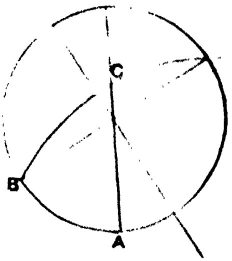


FIG. 26.

101. Prob. 2.—*To project a right angled spherical when the hypotenuse and one side are given.*

SOLUTION.—Using the common notation, let *c* represent the *known* side. Drawing the primitive circle and the conjugate* diameters **BB′**, **NN′**, lay off **BA** = *c*, and draw the diameter **AA′**. The projection of *b* will lie in **AA′**, and the projection of the vertex **C** will fall somewhere in this line. Now the arc *a* lies in a semi-circumference passing through **B** and **B′**. Conceive this semi-circumference to revolve on **BB′**, as an axis, till it coincides, first, with **BNB′**, and then with **BN′B′**. The point **C** will trace the semi-circumference of a small circle perpendicular to the primitive circle, and whose projection is *dd′*. Hence, making **B***d* = **B***d′* = *a*, and drawing *dd′*, the projection of the vertex **C** lies at the same time in **AA′** and *dd′*, and is therefore at their intersection. Passing an arc of a circle (strictly an ellipse) through **BCB′**, we have **ABC**, the projection desired.

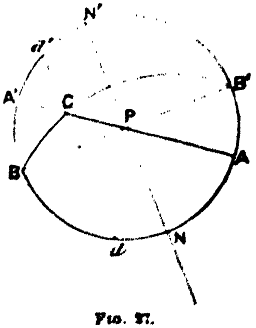


FIG. 27.

---

* Two diameters of a circle which are at right angles to each other are called

QUERIES.—Will a solution of this problem be possible for all values of a and c? If a had been greater than c in the above case, would the solution have been possible? Will dd' and AA′ always intersect, whatever may be the relative values of a and c?

EXAMPLES.

1. Having given $c = 75°$, and $a = 64°$, to project the triangle.

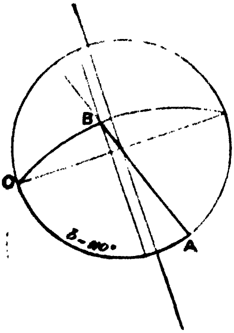


FIG. 28.

2. Having given $c = 45°$, and $a = 136°$, to project the triangle.

3. Having given $b = 110°$, and $a = 85°$, to project the triangle. See *Fig.* 28.

4. Having given $b = 110°$, and $a = 120°$, to project the triangle.

5. Having given $c = 90°$, and $a = 75°$, or $a = 120°$, to project the triangle.

QUERY.—If one side of a right angled spherical triangle is 90°, what must the hypotenuse be? Why?

102. Prob. 3.—*To project a right angled spherical triangle when an oblique angle and the hypotenuse, or the oblique angle and the adjacent side are given.*

SOLUTION.—Draw the primitive circle and the conjugate diameters BB′, NN′ as usual. To construct the given angle B, we observe that this angle is measured by an arc of the great circle which is projected in NN′. Hence, lay off Nd = Nd' = B, and draw dd'; then is NO the projection of the arc which measures B, and the projection of a lies in the arc passing through BOB′.[*] Having the angle B projected, if the hypotenuse a is the other given part, find the projection of C by taking Bc = Bc' = a, and drawing cc'. Complete the projection by drawing AC through P. When the adjacent side c is given, take BA = c, and draw AC as before. [The student will be able to give the reasons.]

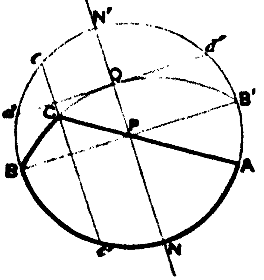


FIG. 29.

103. SCH.—As OP is the cosine of the angle ABC, the point O may be found by measuring from **P** (towards **N** if **B** $< 90°$, and towards **N′** if **B** $> 90°$) a distance equal to the natural cosine of **B**.

---

[*] As has been remarked, this arc is really a semi-ellipse. This fact, together with the the correct projection of the hypot

Query.—Is the solution of this problem possible for all values of the hypotenuse or adjacent side, and the angle?

EXAMPLES.

1. Having given **B** $= 65°$, and the hypotenuse $a = 120°$, to project the triangle. See *Fig.* 30.

2. Having given **C** $= 45°$, and the adjacent side $b = 50°$, to project the triangle.

Sug's.—Project the angle **C** as the angle **B** of the preceding, and lay off $b = 50°$ from its vertex on the circumference of the primitive circle.

3. Having given **C** $= 170°$, and hypotenuse $a = 160°$, to project the triangle.

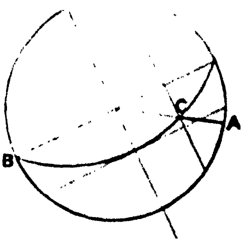


Fig. 30.

4. Having given **B** $= 150°$, and $c = 40°$, to project the triangle.

104. Prob. 4.—*To project a right angled spherical triangle when an oblique angle and side opposite are given.*

Solution.—Project the given angle at **B**, *Figs.* 31, 32, as in the last problem. Then, from any point in the circumference of the primitive circle, as **N**, take **NO′**, in the diameter passing through that point, equal to the projection of the given side. (This is done by taking **N***d* = **N***d′* = b, as in *Prob.* 1, and drawing dd'.) Now, with **P** as a centre, and radius **PO′**, describe a circumference cutting **BOB′**. One extremity of the given side b will be projected in this circumference, since this circumference contains the projections of all the points in the surface of the hemisphere which are at a distance b from the circumference **BNB′N′**. But the vertex **C** is also projected in the

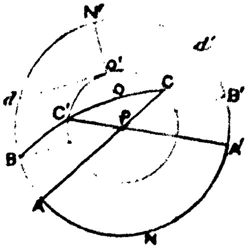


Fig. 31.

---

ence, belong to a treatise on Conic Sections. In this case, **BB′** and **2OP** are the axes ellipse, and the curve can be constructed by taking **BP** as a radius, and striking arcs from **O** as a centre, cutting **BB′**. These intersections are the foci of the required ellipse. Then take a string equal in length to **BB′**, and, fastening its ends to the foci, place a pencil against the string, and keeping the string tight, carry the pencil around the curve.

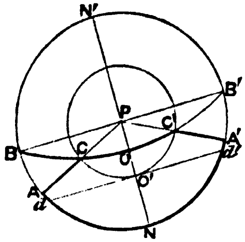


FIG. 32.

arc BOB'; hence, it must be at the intersections C, C'. Drawing the projections of the perpendiculars CA, and C'A', the projection is complete.

QUERIES.—When will there be two triangles fulfilling the given conditions? When but one? When none? When there is but one triangle what kind of a triangle is it? If B > 90°, must *b* be greater or less than B in order to have two solutions? If B < 90°, how is it? If B > 90°, can *b* be less? If B < 90°, can *b* > 90°?

EXAMPLES.

1. *Given* C = 120°, *c* = 150, *to project the triangle. See Fig.* 33.

2. *Given* C = 80°, *c* = 60°, *to project the triangle. See Fig.* 34.

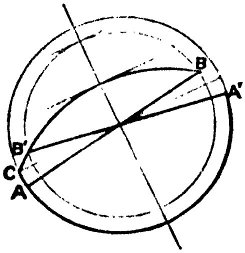


FIG. 33.

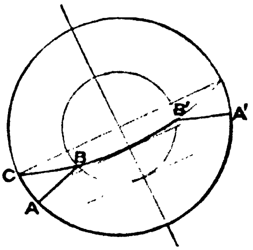


FIG. 34.

3. Given B = 70°, *b* = 70°, to project the triangle. See *Fig.* 35.

4. Given B = 64°, *b* = 75°, to project the triangle. See *Fig.* 36.

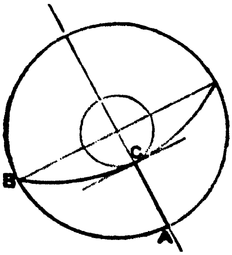


FIG. 35.

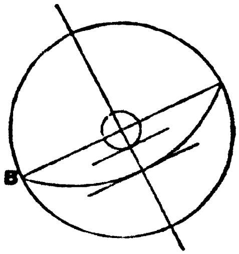


FIG. 36.

5. Given B = 160°, *b* = 110°, to project the triangle.

105. SCH.—When the given parts are the two oblique angles, the projection is most readily effected by first computing one of the sides. The projection in this case will be considered in connection with the numerical solution of the case, in the next section.

PROJECTION OF OBLIQUE ANGLED SPHERICAL TRIANGLES.

106. Prob. 1.—*To project a spherical triangle when two sides and the included angle are given.*

SOLUTION.—Let a, c, and **B** denote the given parts. Project the given angle **B** at some point in the circumference of the primitive circle, as **B**. Lay off one of the given sides, as c, from **B** on **BNB′**. Let **BA** = c. Determine the extremity **C** of the projection of the other given side a, as in *Prob.* 2, etc., and drawing the diameter **AA′**, pass the arc through **ACA′**; **BCA** is the projection sought.

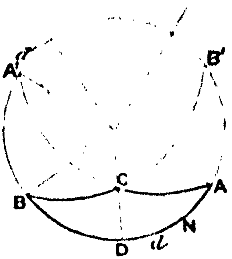


FIG. 37.

QUERY.—Is this projection always possible, what ever the relative magnitude of the given parts?

EXAMPLES.

1. Given **A** = 130°, c = 85°, and b = 100°, to project the

2. Given **C** = 40°, a = 37°, and b = 80°, to project the triangle.

107. Prob. 2.—*To project a spherical triangle when two sides and an angle opposite one of them are given.*

SOLUTION.—Let the given parts be a, b, and **B**. Make the projection upon the plane of the *unknown side* c. Thus, drawing the primitive circle and the conjugate diameters **BB′**, **NN′**, conceive c as projected from **B** on the arc **BNB′**, and project the given angle **B** as in preceding problems. On the arc **BB′**, take **BC** = the projection of the given *adjacent* side a. To determine the projection of the opposite side b, describe a circle about **P**, as a centre, with a radius **PC**. Through **C** draw **PD**, and taking **D***d* = **D***d′* = b draw *dd′*. Through the intersections o, o' of *dd′* with the circumference of the small circle, draw the radii **PA**, **PA′**. Finally, passing arcs through the points **A**, **C**, **A″**, and **A′**, **C**, **A‴**, **BAC**, and **BA′C** are the projections of triangles which fulfill the given conditions. The projections of **B** and

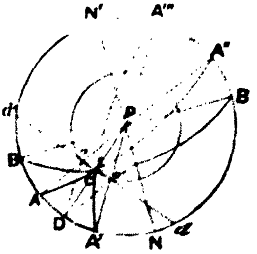


FIG. 38.

a were made upon principles previously established; and it only remains to show that AC, and A′C are projections of *b*. Since by construction DL is the projection of an arc equal to *b*, the projections of arcs of great circles connecting D with *o* and *o′* are projections of arcs equal to *b*. But the figure D*o*P = ACP, and D*o′*P = A′CP; therefore AC = A′C = D*o* = the projection of *b*.

108. Sch.—It is evident that this problem has *one* solution, *two* solutions, or *no* solution, according to the value of *b* as compared with *a* and B. Thus, if the projection of *b* = DC, *o* and *o′* coincide, there is but *one* solution, and the triangle is right angled at A (which in this case falls at D). Also, if the projection of *b* is intermediate in value between DC and *only one* of the arcs BC, B′C, there is only *one* solution. If, however, as in the figure given, the projection of *b* is intermediate in value between DC and *both* BC and B′C, there are *two* solutions. Finally, if *b* is given of such value that the chord *dd′* does not touch the small circle, there is *no* solution. The latter case occurs when B < 90°, if the projection of *b* < DC; and when B > 90°, if the projection of *b* > DC, as will appear from *Figs.* 38, 39.

We may observe, also, that there are *two* solutions when *o* and *o′* both fall on the same side of BB′ as *c*; *one* solution when *o* and *o′* coincide, and when they fall on opposite sides of BB′; and *no* solution when *o* and *o′* are imaginary, *i. e.*, when *dd′* does not touch the small circle, or when both fall on the opposite side of BB′ from *c*.

EXAMPLES.

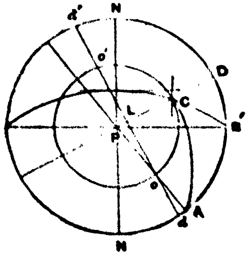


FIG. 39.

1. Given B = 110°, *a* = 120°, and *b* = 83°, to project the triangle. See *Fig.* 39.

2. Given B = 110°, *a* = 120°, and *b* = 130°, to project the triangle.

3. Given C = 64°, *a* = 120°, and *c* = 75°, to project the triangle.

4. Given C = 80°, *b* = 60°, and *c* = 40°, to project the triangle.

5. Given C = 112°, *b* = 75°, and *c* = 150°, to project the triangle.

109. Prob. 3.*—To project a spherical triangle when the three are given.*

Solution.—Drawing the primitive circle and the conjugate diameters, as usual, take **B**A $= c$, the side on the plane of which it is proposed to project the triangle. Take **B**e = **B**$e' = a$ and draw ee'; then as before shown, the projection of the vertex **C** lies in ee'. In like manner taking **A**d = **A**$d' = b$, and drawing dd' the projection of **C** lies in dd'. The intersection of the chords ee' and dd' is therefore the projection of the vertex **C**. Finally, passing arcs through **ACA′** and **BCB′**, the projection is complete.

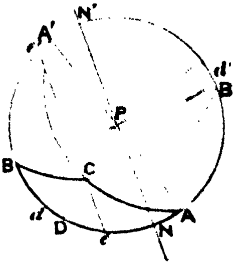


Fig. 40.

Queries.—How does the projection show the impossibility when the sum of the three sides is greater than 360°? How when the sum of two sides is less than the third side?

EXAMPLES.

1. Given $a = 100°$, $b = 80°$, and $c = 68°$, to project the triangle.

2. Given $a = 108°$, $b = 120°$, and $c = 25°$, to project the triangle. See *Fig.* 41.

3. Given $a = 120°$, $b = 65°$, and $c = 40°$, to project the triangle.

4. Given $a = 150°$, $b = 140°$, and $c = 170°$, to project the triangle.

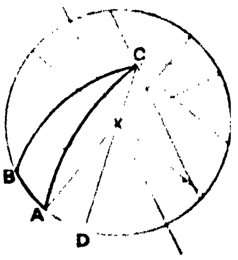


Fig. 41.

SECTION I.

SOLUTION OF RIGHT ANGLED SPHERICAL TRIANGLES.

110. Spherical Trigonometry treats of the relations between the trigonometrical functions of the sides and angles of spherical triangles, and of the solution of such triangles by means of these relations.

111. Sch.—In the present treatise we shall confine our attention none of whose sides or angles exceed 180°.

?. The Six Parts of a spherical triangle are the three sides and the three angles: any three of these being given, the others may be found.

3. Sch.—The last statement involves the assertion that the three angles of a spherical triangle determine the sides, whereas we are accustomed to say in *Plane* Trigonometry, that, at least *one* given part of a plane triangle must be a *side*, in order that the triangle may be determined. There is really no such difference as these two statements imply. For example, if we have the angles of a plane triangle given, we know the ratios of the sides to each other, since the sides are to each other as the sines of the angles opposite; but we cannot determine the *absolute values* of the sides. This is in accordance with the statement that mutually equiangular plane triangles are *similar* figures (not necessarily *equal*). Now these are exactly the facts in the case of spherical triangles, *if we do not limit them to the same or equal spheres*. Thus, the angles of a spherical triangle being given as 48° 30′, 125° 20′, and 62° 54′, we solve the triangle by the rules of spherical trigonometry and find that the sides are 56° 39′ 30″, 114° 29′ 58″, and 83° 12′ 6″. But, so long as the radius of the sphere is unknown, these results are merely the *relative* values of the sides, not their absolute lengths. Moreover, consider two concentric spheres whose radii are *m* and *n*. Now, any triangle being constructed on the one, if planes are passed through its sides intersecting at the common centre, their intersection with the surface of the other sphere will form a triangle mutually equiangular with the first, and any one side of the one triangle is to the corresponding side of the other, as the radii of the spheres; hence the homologous sides are proportional. We see, therefore, that to determine absolutely a spherical triangle, it is necessary to know one of the sides in linear extent as well as angular measure, or, what is equivalent, the radius of the sphere must be known.

114. *The Species* of a part of a spherical triangle is determined by its relation to 90°. Two parts, both greater or both less than 90°, are of the *same species*; two parts, one of which is greater and the other less than 90°, are of *different species*. Thus, two parts which are 58°, and 63°, respectively, are of the same species; two which are 160°, and 115°, are of the same species; two which are 120°, and 48°, are of different species.

Napier's Five Circular Parts are the two sides of a *right angled* spherical triangle about the right angle, and the *complements* of the hypotenuse and of the oblique angles. These terms are merely conventional, and are applied exclusively to right angled triangles.

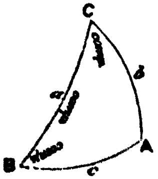


Fig. 42.

Ill.—In a spherical triangle **ABC**, right angled at **A**, the sides *b* and *c*, the complement of hypotenuse *a*, and the complements of the angles **B** and **C**, are the circular parts. We may designate them *b*, *c*, comp *a*, comp **B**, and comp **C**.

[It is essential that this nomenclature and the statements of the two following paragraphs be clearly understood, and firmly fixed in the mind, in order that the phraseology of the fundamental rules may be intelligible.]

3. When five things occur in succession, as it were in a circle, like the circular parts b, c, comp B, comp a, and comp C, in the preceding figure (no account being made of A), it will be observed, that, taking any three at pleasure, one of the three may always be selected which lies adjacent to each of the others, or separated from each of them. Of the three parts thus considered, the ***Middle Part*** is the one which has the other two adjacent to or separated from it; while the latter are called the ***Extremes***, *adjacent* or *opposite*, as the case may be.

ILL'S.—Let the three parts under consideration be comp a, b, and c; comp a is the middle part, and b and c are the *opposite* extremes. If b, c, and comp C are under consideration, b is the middle part, and c and comp C are the *adjacent* extremes.

117. SCH.—In solving a right angled spherical triangle, there are always three parts under consideration at once, viz., the two given parts and the part sought, no mention being made of the right angle. Now, the first question to be settled in order to a solution is, *Which of the three parts under consideration is the middle part, and are the extremes opposite or adjacent?* Beginners are very liable to make mistakes by failing to use the *complements* of the proper parts; or by not correctly distinguishing the middle part, and the character of the extremes, as opposite or adjacent. The student should practise upon such simple exercises as the following until the questions can be answered *instantly*, and ***without***

EXERCISES.

1. In *Fig.* 42, given a and c, to find C. What are the circular parts under consideration? Which is the middle part? Are the extremes adjacent or opposite?

Answers.—The circular parts are comp a, c, and comp C. c is the middle part, and the extremes are opposite.

2. Having C, and a given, to find b. What are the circular parts? Which is the middle part? Are the extremes adjacent or opposite?

3. Having c, and b given, to find D. What are the circular parts? Which is the middle part? Are the extremes adjacent or opposite?

4. Having a, and b given, to find C. What are the circular parts? Which is the middle part? Are the extremes adjacent or opposite?

5. Having B and C given, to find b. What are the circular parts? Which is the middle part? Are the extremes adjacent or

6. What are the opposite extremes when b is the middle part? What the adjacent extremes? Which are the opposite and which the adjacent extremes when c is the middle part? When comp B is the middle part? When comp C is the middle part? When comp a is the middle part?

7. What part is middle part to comp C and c as adjacent extremes? As opposite extremes?

Ans., b, and *none*.

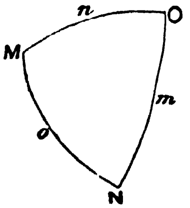


Fig. 43.

8. In *Fig.* 43, M being the right angle, what are the circular parts? Given O and m, to find o. What are the circular parts under consideration? Are the extremes adjacent or opposite?

9. What are the opposite extremes to comp O? What the adjacent? To comp m? To o? To n?

NAPIER'S RULES.

118. Rule I. Prop.—*In any right angled spherical triangle, the sine of the middle part equals the product of the cosines of the opposite extremes.*

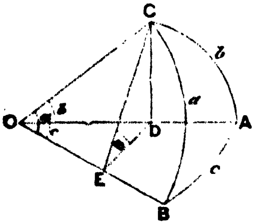


Fig. 44.

DEM.—In the spherical triangle BAC, right angled at A, taking b, c, comp B, comp C, and comp a in succession as middle parts, we are to prove that

$\sin b = \cos(\text{comp } a) \times \cos(\text{comp B})$, or $\sin b = \sin a \sin \text{B}$; (1)

$\sin c = \cos(\text{comp } a) \times \cos(\text{comp C})$, or $\sin c = \sin a \sin \text{C}$; (2)

$\sin(\text{comp B}) = \cos b \times \cos(\text{comp C})$, or $\cos \text{B} = \cos b \sin \text{C}$; (3)

$\sin(\text{comp C}) = \cos c \times \cos(\text{comp B})$, or $\cos \text{C} = \cos c \sin \text{B}$ (4)

$\sin(\text{comp } a) = \cos b \times \cos c$, or $\cos a = \cos b \cos c$. (5)

To demonstrate these relations, let O be the centre of the sphere, and draw the radii OA, OB, and OC. The angles BOC, AOC, and AOB, are measured respectively by a, b, and c, the sides of the triangle; hence these angles at the centre and their measuring arcs may be used interchangeably. From one of the oblique angles, as C, let fall a perpendicular upon the radius OA. From the foot of this perpendicular draw DE perpendicular to OB, and join C and E. Now CDE is a right angle (PART II., *426*), CE is perpendicular to OB (PART II., *399*), and DEC is the measure of angle B of the triangle (PART II., *558*).

$$CD = \sin b,\ OD = \cos b,\ CE = \sin a, \text{ and } OE = \cos a.$$

From the triangle CDE, right angled at D, we have

$$CD = CE \times \sin CED, \quad \text{or} \quad \sin b = \sin a \sin B. \qquad (1)$$

Generalized, (1) becomes, *The sine of either side about the right angle = the sine of the hypotenuse into the sine of the angle opposite the side.* Hence, from analogy to (1), we may write

$$\sin c = \sin a \sin C. \qquad (2)$$

Or (2) may be proved in the same manner as (1), by letting fall from B a perpendicular upon OA, from its foot drawing DE perpendicular to OC, and joining E and B. Then $BD = \sin c$, $OD = \cos c$, $BE = \sin a$, $OE = \cos a$, and angle BED = angle C. From the triangle BDE, we have

$$BD = BE \times \sin BED, \quad \text{or} \quad \sin c = \sin a \sin C. \qquad (2)$$

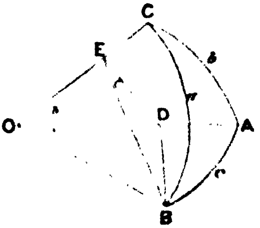


FIG. 45

To prove (3), we have from the triangle CDE, Fig. 44,

$$\cos CED = \frac{ED}{CE}, \quad \text{or} \quad \cos B = \frac{ED}{\sin a}.$$

But from triangle OED, right angled at E, $ED = OD \times \sin DOE = \cos b \sin c =$ [from (2)], $\cos b \sin a \sin C$. Substituting this value of ED, we have

$$\cos B = \frac{\cos b \sin a \sin C}{\sin a} = \cos b \sin C. \qquad (3)$$

We may write (4) from (3) by analogy, as (2) was from (1); or, better, let the student produce it from Fig. 45, as (3) was produced from Fig. 44.

Finally, to produce (5), consider the triangle ODE, in either figure, right angled at E. This gives

$$OE = OD \times \cos DOE, \quad \text{or} \quad \cos a = \cos b \cos c. \qquad (5)$$

119. *Rule 2.* *Prop.*—*In any right angled spherical triangle, the sine of the middle part equals the product of the tangents of the adjacent extremes.*

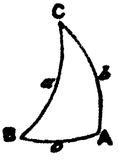


FIG. 46.

DEM.—In the spherical triangle BAC, right angled at A, taking b, c, comp B, comp C, and comp a, in succession as middle parts we are to prove that,

$$\sin b = \tan c \times \tan(\text{comp C}), \text{ or } \sin b = \tan c \cot C; \qquad (1)$$
$$\sin c = \tan b \times \tan(\text{comp B}), \text{ or } \sin c = \tan b \cot B; \qquad (2)$$
$$\sin(\text{comp B}) = \tan c \times \tan(\text{comp } a), \text{ or } \cos B = \cot a \tan c; \qquad (3)$$
$$\sin(\text{comp C}) = \tan b \times \tan(\text{comp } a), \text{ or } \cos C = \cot a \tan b; \qquad (4)$$
$$\sin(\text{comp } a) = \tan(\text{comp B}) \times \tan(\text{comp C}), \text{ or } \cos a = \cot B \cot C \cdot \qquad (5)$$

Taking the *formulæ* of Rule 1st, and in the second member of each substituting the value of each factor as found in some other of the set, we readily write the following

$$\sin b = \sin a \sin B = \frac{\sin c \cos C}{\sin C \cos c} = \frac{\sin c}{\cos c} \times \frac{\cos C}{\sin C} = \tan c \cot C; \qquad (1)$$
$$\sin c = \sin a \sin C = \frac{\sin b \cos B}{\sin B \cos b} = \frac{\sin b}{\cos b} \times \frac{\cos B}{\sin B} = \tan b \cot B; \qquad (2)$$
$$\cos B = \cos b \sin C = \frac{\cos a \sin c}{\cos c \sin a} = \frac{\cos a}{\sin a} \times \frac{\sin c}{\cos c} = \cot a \tan c; \qquad (3)$$
$$\cos C = \cos c \sin B = \frac{\cos a \sin b}{\cos b \sin a} = \frac{\cos a}{\sin a} \times \frac{\sin b}{\cos b} = \cot a \tan b; \qquad (4)$$
$$\cos a = \cos b \cos c = \frac{\cos B \cos C}{\sin C \sin B} = \frac{\cos B}{\sin B} \times \frac{\cos C}{\sin C} = \cot B \cot C. \qquad (5)$$

Q. E. D.

120. Sch. 1.—It will be a good exercise for the student to demonstrate Rule 2d from the annexed figures, as Rule 1st was from *Figs.* 44 and 45. The 5th is not as readily obtained from the figure as the others. The stu-

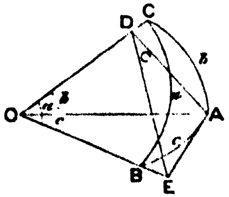


Fig. 47.

Fig. 48.

lowing relations, some in one figure, and some in the other.

$$\text{Cos } a = \frac{OD}{OE} = \frac{\cos b}{\sec c} = \cos b \cos c. \text{ But, } \cot C = \frac{AD}{AE} = \frac{\sin b}{\tan c}, \text{ and } \cot B = \frac{AD}{AE}$$

$$= \frac{}{\tan b}; \text{ whence, } \cot B \cot C = \frac{\sin b}{\tan b}\,\frac{\sin c}{\tan c} = \cos b \cos c. \therefore \cos a = \cot B \cot C.$$

121. Sch. 2.—It is of much importance, especially for the purposes of Spherical Astronomy, that the student observe that the relations expressed in the above formulæ, and in fact all the relations between the sides and angles of spherical triangles, are also the relations between the facial and diedral angles of triedrals. Thus, if a, b, and c represent the facial angles, and A, B, and C the opposite diedrals, all these relations can be established, and in exactly the same manner as above, *without any allusion to the spherical triangle.* [The student should do it.]

DETERMINATION OF SPECIES.

122. In the solution of spherical triangles the determination of the species of a part sought becomes of essential importance, since *any* part of such a triangle may have any value between 0° and 180°. Hence, when we have learned the numerical value of any function of a part, we have yet to determine whether the part itself is less or greater than 90°, *i. e.*, the species of the part. This may always be effected by some one of the following propositions.

123. Prop.*—If the part sought is found in terms of its cos, tan, or cot, its species can be determined by the algebraic signs of the functions in the formula used.*

a.—In each of the *formulæ* arising from the application of Napier's rules, there are three functions, the arcs corresponding to two of which are always known, hence the algebraic signs of their functions are known, and the signs of these two determine the sign of the third or unknown function. Now, when a cos, tan, or cot is +, the corresponding angle is less than 90° (if less than 180°); and when one of these functions is , the corresponding angle is greater than 90°; *i. e.*, in a triangle, it is between 90° and 180°.

124. When the part sought is found in terms of its sine, the species cannot be determined by the signs of the formula, since the part being less than 180° its sine is always +. The three following propositions dispose of such cases.

125. Prop.*—An oblique angle of a right angled spherical triangle and its opposite side are always of the same species.*

I.—From Napier's first rule we have, $\cos B = \cos b \sin C$. But $\sin C$ is necessarily +; therefore, $\cos B$ and $\cos b$ always have the same sign, and B and b are of the same species. In like manner, we see from $\cos C = \cos c \sin B$, that C and c are of the same species. Q. E. D.

3. Prop.*—When the hypotenuse of a right angled spherical triangle is less than 90°, the other two sides (and* *oblique angles) are of the same species with each other. But when the hypotenuse is greater than 90°, the other two sides (and consequently the oblique angles) are of different species from each other.*

DEM.—From Napier's first rule we have, $\cos a = \cos b \cos c$. Now, if $a < 90°$, $\cos a$ is $+$; hence $\cos b$ and $\cos c$ must have *like* signs, and b and c be both less or both greater than 90° But if $a > 90°$ (and less than 180°, as it is), $\cos a$ is $-$; hence, $\cos b$ and $\cos c$ must have *different* signs, and b and c be one greater and the other less than 90°. Finally, since the oblique angles are of the same species as their opposite sides, they are of the same species with each other when $a < 90°$, and of different species from each other when $a > 90°$.

127. Prop.—*When a side and its opposite angle are given in a right angled spherical triangle, there is* NO *solution if the sine of the side is greater than the sine of its opposite angle; there is* ONE *solution and the triangle is bi-rectangular if the sine of the side equals the sine of its opposite angle; and there are* TWO *solutions if the sine of the side is less than the sine of its opposite angle.*

DEM.—We have $\sin b = \sin a \sin B$, or $\sin a = \frac{\sin b}{\sin B}$. Now, $\sin b > \sin B$ makes $\sin a > 1$, which is manifestly impossible. $\sin b = \sin B$ makes $b = B$, since they are of the same species. But when the arc included by the sides of an oblique angle of a right angled spherical triangle is equal to the angle, the vertex of the angle is the pole of the arc. Hence, in this case the other sides of the triangle are each 90°. Finally, if $\sin b < \sin B$, a has two values, one greater and the other less than 90°. Hence there are two triangles.

SCH.—These relations between an angle and its opposite side may be observed directly from a figure. When $B < 90°$, the measure of it, that is $b = B$, is the greatest included perpendicular which can be drawn to one side of the lune BAB′C. Hence, in this case, b cannot exceed B, which implies that $\sin b$ cannot be $> \sin B$, as when the arcs are less than 90°, the greater arc has the greater sine. If $\sin b = \sin B$, $b = B$, and $BA = BC = 90°$, and the side b can occupy but one position in the lune, thus giving rise to but one triangle BAC, which satisfies the conditions (or two equal triangles BAC and B′AC). If $b < B$, which, when B is less than 90°, implies that $\sin b < \sin B$, the side can occupy two positions in the lune, b' and b'' giving rise to two triangles, BA′C′ and BA″C″, both of which fulfill the conditions.

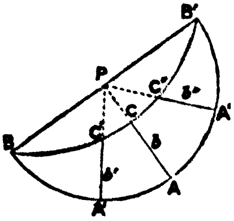


FIG. 49.

Again, when $B > 90°$, the measure of it, *i. e.*, $b = B$, is the *least* included perpendicular that can be drawn to one side of the lune. Hence, in this case we cannot have $b < B$, which implies that $\sin b$ cannot be $> \sin B$, since the greater arc has the less sine. If $\sin b = \sin B$, $b = B$, and $BA = BC = 90°$, and the side b can occupy but one position in the lune, thus giving rise to but one triangle BAC, which satisfies the conditions (or two equal triangles BAC and $B'AC$). If $b > B$, which implies that $\sin b < \sin B$, the side b can occupy two positions in the lune, as b' and b'', thus giving rise to two triangles $BA'C'$, and $BA''C''$, both of which satisfy the conditions.

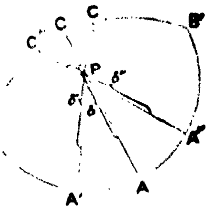


FIG. 50.

EXERCISES.

1. In a right angled spherical triangle BAC, A being the right angle, $B = 80° 40'$, and $a = 105° 34'$, to project the triangle and compute the other parts.

PROJECTION.*—Projecting the triangle upon the plane of the side c we have, BCA, *Fig.* 51. [The student should give the process.]

ON.—It is immaterial which of the required parts we seek first. We will seek c. Now the three circular parts under consideration are c, comp a, and comp B. Comp B is middle part, and the extremes are adjacent; hence, by Napier's second rule we have,

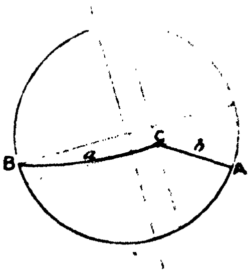


FIG. 51.

$$\cos B = \tan c \cot a.$$

or
$$\tan c = \frac{\cos B}{\cot a} = \frac{\cos 80°\ 40'}{\cot 105°\ 34'}.$$

Now $\cos 80°\ 40'$ is $+$, and $\cot 105°\ 34'$ is $-$. Therefore $\tan c$ is $-$, and $c >$

Computing by logarithms,

$$\begin{aligned} \log \cos\ 80°\ 40' &= 9.209992 \\ -\log \cot\ 105°\ 34' &= 9.444947 \\ = \log \tan c\ \ \ &= 1.765045 \\ \text{Add 10 for tab. tan}\ \ &\ \ \underline{10.\qquad\quad} \end{aligned}$$

$$\therefore c = 149°\ 47'\ 37''.$$

---

* It is recommended that the projection be given
It is an excellent exercise, and give

To find b, $\sin b = \sin a \sin B = \sin 105^\circ\ 34' \times \sin 80^\circ\ 40'$. This makes b known by means of its sine, whence the signs of the formula do not determine the species of b. But b is of the same species as B (*124*), and therefore less than 90°.

Computing by logarithms,

$$\begin{aligned} \log \sin 105^\circ\ 34' &= 9.983770 \\ + \log \sin 80^\circ\ 40' &= 9.994212 \\ \text{Deducting } 10 &= 9.977982 = \log \sin b. \therefore b = 71^\circ\ 54'\ 33''. \end{aligned}$$

To find C, $\cos a = \cot B \cot C$, or $\cot C = \dfrac{\cos a}{\cot B} = \dfrac{\cos 105^\circ\ 34'}{\cot 80^\circ\ 40'}$. Whence $\cot C$ is $-$, and $C > 90^\circ$.

Computing by logarithms,

$$\begin{aligned} \log \cos 105^\circ\ 34' &= 9.428717 \\ - \log \cot 80^\circ\ 40' &= 9.215780 \\ \text{Adding } 10 &= 10.212937 = \log \cot C. \therefore C = 148^\circ\ 30'\ 54''. \end{aligned}$$

SCH.—It is expedient to find each part directly from the parts given in the example, in order that an error in finding one may not extend itself through the whole solution.

2. Given $a = 86^\circ\ 51'$, and $B = 18^\circ\ 03'\ 32''$, to project the triangle and compute the other parts.

$c = 86^\circ\ 41'\ 14''$, $b = 18^\circ\ 01'\ 50''$, $C = 88^\circ\ 58'\ 25''$.

3. Given $b = 155^\circ\ 27'\ 54''$, and $c = 29^\circ\ 46'\ 08''$, to project the triangle and compute the other parts. See *Fig.* 52.

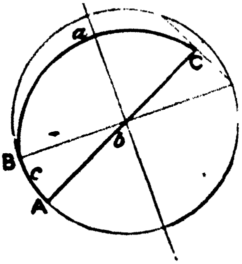


Fig. 52.

$a = 142^\circ\ 09'\ 13''$, $C = 54^\circ\ 01'\ 16''$, $B = 137^\circ\ 24'\ 21''$.

4. Given $c = 73^\circ\ 41'\ 35''$, and $B = 99^\circ\ 17'\ 33''$, to project the triangle and compute the other parts.

$C = 73^\circ\ 54'\ 46''$, $b = 99^\circ\ 40'\ 30''$, $a = 92^\circ\ 42'\ 17''$.

5. Given $B = 47^\circ\ 13'\ 43''$, and $C = 126^\circ\ 40'\ 24''$, to project the triangle and compute the other parts.

$b = 32^\circ\ 08'\ 56''$, $a = 133^\circ\ 32'\ 26''$, $c = 144^\circ\ 27'\ 03''$.

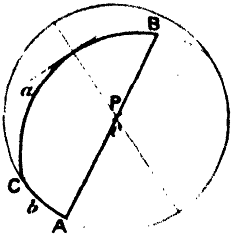


FIG. 53.

PROJECTION.—In order to project this case, *i. e.*, when the two oblique angles are given (***105***), it is most convenient to compute the base before projecting. It is also expedient, when *two angles* are given, to project the *larger* at a point in the circumference of the primitive circle, as at **C**, especially if the smaller be quite small. In this case, projecting the angle **C** at **C**, Fig. 53, *conceive* **BA** as drawn through **P** (or, if desired, sketch it hypothetically), and then compute b, from the relation $\cos \mathrm{B} = \sin \mathrm{C} \cos b$, or $\cos b = \dfrac{\cos \mathrm{B}}{\sin \mathrm{C}}$. Having found $b = 32° \, 08' \, 56''$, take **CA** $= b$, and draw **AB** through **P**.

6. Given $\mathrm{B} = 100°$, and $b = 112°$, to project the triangle and compute the other parts.

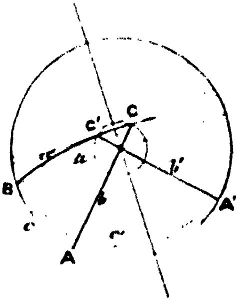


FIG. 54.

PROJECTION.—See Fig. 54.

NUMERICAL SOLUTION.—*To find* c, we have

$$\sin c = \tan b \cot \mathrm{B} = \tan 112° \cot 100°.$$

Computing by logarithms,

$$\begin{aligned} \log \tan 112° &= 10.393590 \\ + \log \cot 100° &= 9.246319 \\ \text{Rejecting } 10 &= 9.639909 = \log \sin c. \end{aligned}$$

$\therefore c = 25° \, 52' \, 33''.4$, or $154° \, 07' \, 26''.6$, *i. e.*, **BA**, or **BA'**.

To find a, we have

$$\sin b = \sin a \sin \mathrm{B}; \text{ whence } \sin a = \frac{\sin b}{\sin \mathrm{B}} = \frac{\sin 112°}{\sin 100°}.$$

Computing by logarithms,

$$\begin{aligned} \log \sin 112° &= 9.967166 \\ - \log \sin 100° &= 9.993351 \end{aligned}$$

Adding 10 $= 9.973815 = \log \sin a$. $\therefore a = 70° \, 18' \, 10''.7$, or $109° \, 41' \, 49''.3$, *i. e.*, **BC'**, or **BC**.

To find **C**, we have

$$\cos \mathrm{B} = \cos b \sin \mathrm{C}; \text{ whence } \sin \mathrm{C} = \frac{\cos \mathrm{B}}{\cos b} = \frac{\cos 100°}{\cos 112°}.$$

Computing by logarithms,

$$\begin{aligned} \log \cos 100° &= 9.239670 \\ - \log \cos 112° &= 9.573575 \end{aligned}$$

Adding 10 $= 9.666095 = \log \sin \mathrm{C}$. $\therefore \mathrm{C} = 27° \, 36' \, 56''$ *i. e.*, **BCA**, or **BC'A'**.

Thus we see that each of the two triangles BCA and BC′A′ fulfills the conditions of the problem.

7. Given one side of a right angled spherical triangle 160°, and the opposite angle 150°, to project the triangle and compute the other parts.

Results.—There are two triangles. The other sides of the first are 136° 50′ 23″, and 39° 04′ 51″; and the angle opposite the latter side is 67° 09′ 43″. The corresponding parts of the other triangle are 43° 09′ 37″, 140° 55′ 09″, and 112° 50′ 17″.

8. In the spherical triangle DEF, right angled at E, given an oblique angle 58°, and the side opposite 64°, to project the triangle and compute the other parts.

9. In a right angled spherical triangle given an oblique angle 165°, and the opposite side 112°, to project the triangle and compute the other parts.

10. In a right angled spherical triangle given one side 65° 23′ 12″, and the opposite angle 65° 23′ 12″, to project the triangle and compute the other parts.

11. Given C = 60° 47′ 24″.3, B = 57° 16′ 20″.2, and A = 90°, to project and compute.

$a = 68° 56′ 28″.9$, $c = 54° 32′ 32″.1$, and $b = 51° 43′ 36″.1$.

12. Given $c = 116°$, $b = 16°$, and the included angle 90°, to project and compute.

TRIANGLES.

129. ***A Quadrantal Triangle*** is a spherical triangle one of whose sides is a quadrant, or 90°. Such a triangle is readily solved by passing to its polar, solving it, and then passing back. The polar triangle to a quadrantal triangle, being right angled, is solved by Napier's rules.

Ex. 1. Given $a = 90°$, B $= 75° 42′$, and $c = 18° 37′$, to compute the other parts.

Sol'n.—Representing the supplemental parts of the polar triangle by A′, B′, C′, $a′$, $b′$, and $c′$, we have A′ $= 180° - a = 90°$, $b′ = 180° -$ B $= 104° 18′$, and

$C' = 180° - c = 161° 23'$, from which to find B', a', and c'. This being right angled, we find, by applying Napier's rules, $B' = 94° 31' 21''$, $a' = 76° 25' 11''$, and $c' = 161° 55' 20''$. Hence in the primitive triangle we have $b = 180° - B' = 85° 28' 39''$, $A = 180° - a' = 103° 34' 49''$, and $C = 180° - c' = 18° 04' 40''$.

Ex. 2. Given $a = 90°$, $C = 42° 10'$, and $A = 115° 20'$, to find the other parts.

$$B = 54° 44' 24'', \; b = 64° 36' 40'', \; c = 47° 57' 47''.$$

SECTION II.

OF OBLIQUE ANGLED SPHERICAL TRIANGLES.

130. All cases of oblique angled spherical triangles can be solved by Napier's rules and the following proposition.

131. Prop.—*In any spherical triangle, if a perpendicular be let fall from either vertex upon the opposite side or side produced, the tangent of half the sum of the segments* of that side is to the tangent of half the sum of the other two sides, as the tangent of half the difference of those sides is to the tangent of half the difference of the segments.*

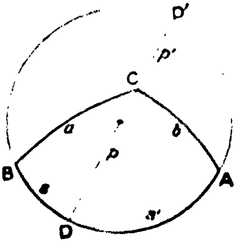


FIG. 55.

DEM.—In the triangle **BAC** let fall the perpendicular p, from **C** upon the opposite side. Let $BD = s$, and $DA = s'$. By Napier's first rule, $\cos a = \cos p \cos s$, and $\cos b = \cos p \cos s'$. Dividing the former by the latter, $\frac{\cos a}{\cos b} = \frac{\cos s}{\cos s'}$; whence, by composition and division,

But by (*61*),

$$\frac{\cos b - \cos a}{\cos a + \cos b} = \tan \tfrac{1}{2}(a + b) \tan \tfrac{1}{2}(a - b),$$

and

$$\frac{\cos s' - \cos s}{\cos s + \cos s'} = \tan \tfrac{1}{2}(s + s') \tan \tfrac{1}{2}(s - s'),$$

---

* When the perpendicular falls without the base, as in Fig. 56, this term is to be un as meaning the distances from the foot of the perpendicular to each extremity of the **BD** and **AD**. This, in fact, is the general statement—applying as well to the case perpendicular falls on the base.

$$\therefore \tan \tfrac{1}{2}(a+b) \tan \tfrac{1}{2}(a-b) = \tan \tfrac{1}{2}(s+s') \tan \tfrac{1}{2}(s-s');$$

or, $\tan \frac{1}{2}(s+s') : \tan \frac{1}{2}(a+b) :: \tan \frac{1}{2}(a-b) : \tan \frac{1}{2}(s-s')$. Q. E. D.

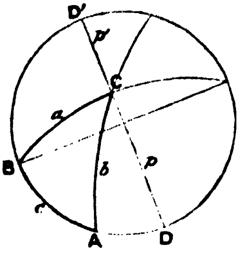


FIG. 56.

132. SCH. 1.—Since from a point in the surface of a hemisphere *two* perpendiculars can always be drawn to the circumference of the great circle which forms its base, and since the feet of these perpendiculars are 180° apart, and no side of a spherical triangle can equal 180°, the foot of one perpendicular will always fall within the base or upon one extremity of it, and the other without the base; or both will fall without the base. If we take the foot of the perpendicular which falls within the base, or the *nearer* one when both fall without, the *sum* of the distances from the foot of the perpendicular to the extremities of the base is always less than 180°, *i. e.*, $s + s' < 180°$. When the perpendicular falls within, $s + s'$ makes up one side of the triangle, and hence is less than 180°. If both perpendiculars fall without, let D, *Fig.* 56, be the foot of the nearer one. Now $DB + BD' = 180°$; but by hypothesis $DA < BD'$, $\therefore DB + DA < 180°$. When $DA = BD'$, $DB + DA = 180°$.

133. SCH. 2.—As in spherical triangles the greater segment is not always adjacent to the greater side, it becomes necessary to determine the position of the segments. This can be done by the signs of the proportion

$$\tan \tfrac{1}{2}(s+s') : \tan \tfrac{1}{2}(a+b) :: \tan \tfrac{1}{2}(a-b) : \tan \tfrac{1}{2}(s-s').$$

1st. Tan $\frac{1}{2}(s+s')$ is always +, since, if D falls in the base, $s + s' < 180°$; and if D falls without, by taking the nearer perpendicular, $s + s'$ is still $< 180°$ (***132***). $\therefore \frac{1}{2}(s+s') < 90°$, and $\tan \frac{1}{2}(s+s)$ is +.

2d. When $a + b < 180°$, $\tan \frac{1}{2}(a+b)$ is +; and when $a + b > 180°$. $\tan \frac{1}{2}(a+b)$ is −.

3d. When $a > b$, $a - b$ is a positive arc less than 180°, hence $\tan \frac{1}{2}(a-b)$ is +; and when $a < b$, $(a - b)$ is a negative arc and less than 180°, hence $\tan \frac{1}{2}(a-b)$ is −.

4th. The signs of these terms being determined, that of $\tan \frac{1}{2}(s-s')$ known. Now, as $\frac{1}{2}(s-s')$ cannot be numerically greater than 90°, tan is + when $s > s'$, and − when $s < s'$.

5th. When $s + s' = 180°$, $\tan \frac{1}{2}(s+s') = \infty$. Now as $a - b < 180°$, $\tan \frac{1}{2}(a-b)$ cannot be ∞, nor can $\tan \frac{1}{2}(s-s') = 0$ when the perpendicular falls without. Hence to make the proportion possible, $\tan \frac{1}{2}(a+b)$ must be ∞, or $a + b = 180°$. In this case we project on the plane of a or b. If $a + b = 180°$, and $a + c = 180°$, we project on the plane of a. If $a + b = 180°$, $a + c = 180°$, and $b + c = 180°$, the triangle is trirectangular.

134. SCH. 3.—If either segment is greater than the whole base, the perpendicular falls without the triangle. In this case the shorter segment lies in an opposite direction from its angle to that considered in the demonstration, and hence is to be considered −; and $s + s'$ is in every case equal to the side upon which the perpendicular is let fall.

135. Prop.—*In a spherical triangle, the sines of the sides are to each other as the sines of their opposite angles.*

Dem.—By Napier's first rule we have from either *Fig.* 55 or *Fig.* 56,

$$\sin p = \sin a \sin \mathsf{B}, \quad \text{and} \quad \sin p = \sin b \sin \mathsf{A}.$$

$$\therefore \sin a \sin \mathsf{B} = \sin b \sin \mathsf{A}, \quad \text{or} \quad \sin a : \sin b :: \sin \mathsf{A} : \sin \mathsf{B}. \quad \text{Q. E. D.}$$

Sch.—This proposition is not introduced here because it is *necessary* for the *solution* of spherical triangles, but because of its essential importance. It is often *convenient* to use it in the solution of a triangle, but never necessary, as will appear hereafter. *It affords a ready method of determining a part* OPPOSITE *a given part, provided the species of the part be determined by other*

SOLUTION OF OBLIQUE ANGLED SPHERICAL TRIANGLES BY NAPIER'S RULES FOR RIGHT ANGLED SPHERICAL TRIANGLES.

136. The examples which arise in the solution of oblique angled spherical triangles are all comprised under the three following problems, each of which consists of two cases:—

1. When the given parts are all adjacent to each other.
2. When two of the given parts are adjacent and one separate.
3. When the given parts are all separate from each other.

137. Prob. 1.—*Given three adjacent parts of an oblique angled spherical triangle, to solve the triangle.*

Case 1st.—*Given two sides and the included angle.*

Solution.—Project the triangle on the plane of *one of the given sides* (**108**), and let fall a perpendicular from the angle opposite upon this side or upon

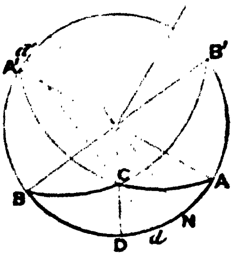
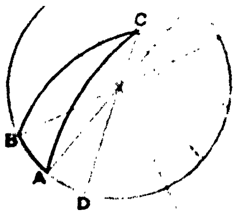


Fig. 57.



this side produced, as the case may be. There are thus formed two right angled triangles, as BDC and DCA, each of which can be solved by Napier's rules, by first solving the one containing the given angle. Thus, in the triangle **BDC** right angled at **D**, a and **B** are supposed known; whence **CD**, **BD**, and the angle **BCD**, can be computed. As **BA** $= c$ is known, the segment **DA** can be found, it being the difference between c and the arc **BD**. When the solution of this triangle gives **BD** $> c$, it is evident that the perpendicular falls without the triangle, which will agree with the projection. Passing to the triangle **ADC**, right angled at **D**, we now know **CD** and **DA**; whence the other parts can be found. Finally, the angle **BCA** of the required triangle = **BCD** + **DCA** when the perpendicular falls within the triangle, and **BCD** − **DCA** *when the perpendicular falls* without.

CASE 2d.—*Given two angles and the included side.*

SOLUTION.—The solution of this case is effected by passing to the polar triangle, projecting and solving it by Case 1st, and then passing back.

138. SCH.—A slight saving of labor is effected by using (***135***) in the solution. Thus, in the triangle **BCD**, compute **CD** and **BD** as before, and (not computing angle **BCD**) then passing to the triangle **DCA**, compute b and **A**. Finally, compute **C** (the entire angle) from the proportion

$$\sin b : \sin c :: \sin \mathrm{B} : \sin \mathrm{C}.$$

139. Prob. 2.—*In an oblique angled spherical triangle, given two parts adjacent to each other and one separated from both of them, to solve the triangle.*

CASE 1st.—*Given two sides and an angle opposite one of them.*

SOLUTION.—Project the triangle on the plane of the *unknown side*, with the given angle at **B**; and let fall the perpendicular from the angle **C** opposite the unknown side. Compute the triangle **BDC**. Having computed this triangle,

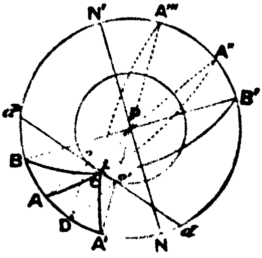


FIG. 59.

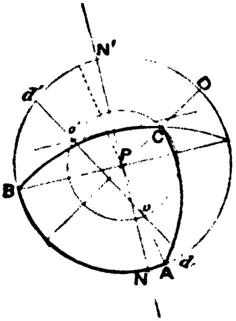


FIG. 60.

the side opposite the given angle, as b, with the perpendicular and the arcs **BC** and **CB'**, *i. e.*, with p, a, and $180° - a$. If $b = p$ there is but *one* solution and the triangle is right angled, **A** falling at **D**. If b is intermediate in value between p and *both* a and $180° - a$, it can occupy two positions as in *Fig.* 59, and there are two solutions. If b is intermediate in value between p and only one of the arcs a or $180° - a$, there is but one solution. When $B < 90$ the perpendicular is *less* than any oblique arc; hence in this case, if $b < p$, there is *no* solution. But if $B > 90°$, the perpendicular is greater than the oblique arcs; hence in this case, if $b > p$, there is *no* solution. [These results should be obtained independently of the results given by the projection, and one be made a check upon the other.] The solution is now completed by computing the parts of **DCA**, and adding or subtracting the segments **BD** and **AD**, and the angles **BCD** and **ACD**, as the case may require.

CASE 2d.—*Given two angles and a side opposite one of them.*

SOLUTION.—Pass to the polar triangle; solve it, and then pass back.*

9. SCH.—The relation established in (*135*) may also be used in the solution of this problem. Thus, having projected the triangle, computed p, and determined whether there are one or two solutions, to find **A**, when a, b, and **B** are given, we have, $\sin b : \sin a :: \sin B : \sin A$. Then computing the third side c (or sides), by means of the right angled triangles **BCD** and **DCA** as before, we may use the proportion (*135*) to find the angle **BCA** and **BCA′**. But the use of this proportion gives no advantage except in cases in which there is only one solution.

141. Prob. 3.—In an obliqu
three parts all separated from each o

CASE 1st.—*Given the sides to find the angles.*

SOLUTION.—Project the triangle on the plane of one of its sides, as c. From the proportion,

$$\tan \tfrac{1}{2}(s + s') : \tan \tfrac{1}{2}(a + b) :: \tan \tfrac{1}{2}(a - b) : \tan \tfrac{1}{2}(s - s'),$$

---

* This case can be projected and solved in a manner altogether similar to the first, without passing to the polar triangle. Thus, let **B**, a, and **A** be the given parts. Project the triangle on the plane of c, as in the figure. Project **B** in the usual way, and make **BC** the projection of a. Through **C** draw **DD′**, and make **BDO** the projection of **A**. Drawing the small circle with radius **PC**, draw diameters through the intersections **O** and **O′**. Then will **A** and **A′** be the vertices of the triangle required. The student may prove that the figures **POD** and **PCA** are equal, and also **PO′D** and **PCA′**, and hence that angle **BDO** = **A** = **A′**.

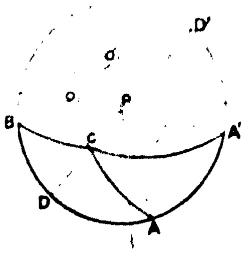


FIG. 61.

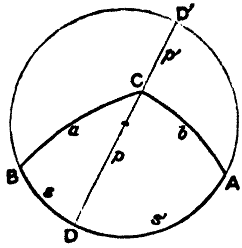


FIG. 62.

find $\frac{1}{2}(s - s')$. Then half the sum of the segments + half the difference gives the greater segment, and half the sum − half the difference gives the less. Determine from the signs of the terms whether s is greater or less than s' : and also determine whether the perpendicular lies within or without the triangle (*134*). Observe that these results correspond to those given by the projection. Finally, in each of the two right angled triangles **BCD** and **DCA**, there are two sides given; whence the angles can be found by Napier's rules. If the perpendicular falls within, **C** = **BCD** + **DCA**, and **A**, of the required triangle = **DAC**. If the perpendicular falls without, **C** = **BCD** − **DCA**, and **A** of the triangle = 180° − **DAC**.

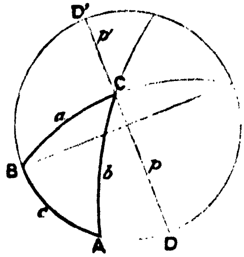


FIG. 63

CASE 2d.—*Given the angles to find the sides.*

SOLUTION —Pass to the polar triangle; solve it, and then pass back.

142. SCH.—Here, again, (*135*) affords a slightly more expeditious solution. Having projected the triangle, found and located the segments, and computed one angle, as **B**, by the methods given above, the other angles may be found from the proportions,

$$\sin b : \sin a :: \sin \mathrm{B} : \sin \mathrm{A},$$

and

$$\sin b : \sin c :: \sin \mathrm{B} : \sin \mathrm{C}.$$

EXERCISES.

1. Given $b = 120° 30' 30''$, $c = 70° 20' 20''$, and **A** = 50° 10′ 10″, to project and solve the triangle.

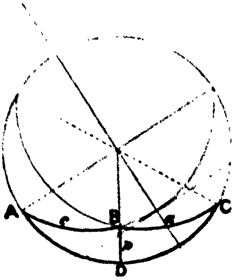


FIG. 64.

PROJECTION.—See *Fig.* 64.

TRIGONOMETRICAL SOLUTION.—1st. *To solve the triangle* **ABD**, in which the two known parts are situated.

(*a*) To find p, $\sin p = \sin c \sin \mathrm{A}$.

$\log \sin 70° 20' 20''$

$+ \log \sin 50° 10' 10'' =$

Rejecting 10 = 9.859241 = log sin p

$\therefore p = 46° 19' 01''$, the species being determined by the opposite angle (*125*). [Observe that the result corresponds with the projection].

(*b*) To find AD, $\cos A = \cot c \tan AD$, or $\tan AD = \dfrac{\cos A}{\cot c}$.

$$\begin{aligned} \log \cos 50^\circ\ 10'\ 10'' &= 9.806532 \\ -\ \log \cot 70^\circ\ 20'\ 20'' &= 9.553016 \end{aligned}$$

Adding 10 = 10.253516 = log tan AD. ∴ AD = 60° 50′ 49″, the species being determined by the signs of the formula.

(*c*) To find angle ABD, $\cos c = \cot A \cot ABD$, or $\cot ABD = \dfrac{\cos c}{\cot A}$.

$$\begin{aligned} \log \cos 70^\circ\ 20'\ 20'' &= 9.526929 \\ -\ \log \cot 50^\circ\ 10'\ 10'' &= 9.921204 \end{aligned}$$

Adding 10 = 9.605725 = log cot ABD. ∴ ABD = 68° 01′ 53″ the species being determined by the signs of the formula.

2d. *To solve the triangle* DBC.

(*a*) To find DC. Since AD < *b*, the foot of the perpendicular falls in the base, and DC = AC − AD = *b* − AD = 120° 30′ 30″ − 60° 50′ 49″ = 59° 39′ 41″.

(*b*) To find *a*, $\cos a = \cos p \cos DC$

$$\begin{aligned} \log \cos 46^\circ\ 19'\ 01'' &= 9.839270 \\ +\ \log \cos 59^\circ\ 39'\ 41'' &= 9.703386 \end{aligned}$$

Rejecting 10 = 9.542656 = log cos *a*. ∴ *a* = 69° 34′ 56″, the species being determined by the signs of the formula.

(*c*) To find C, $\sin p = \sin a \sin C$, or $\sin C = \dfrac{\sin p}{\sin a}$.

$$\begin{aligned} \log \sin p &= 9.859241 \\ -\ \log \sin 69^\circ\ 34'\ 56'' &= 9.971820 \end{aligned}$$

Adding 10 = 9.887421 = log sin C. ∴ C = 50° 30′ 08″, the species being determined by the side opposite.

(*d*) To find angle DBC, $\sin p = \tan DC \cot DBC$, or $\cot DBC = \dfrac{\sin p}{\tan DC}$.

$$\begin{aligned} \log \sin p &= 9.859241 \\ -\ \log \tan 59^\circ\ 39'\ 41'' &= 10.232653 \end{aligned}$$

Adding 10 = 9.626588 = log cot DBC. ∴ DBC = 67° 03′ 36″, the species being determined by the signs of the formula.

Finally, B = ABD + DBC = 68° 01′ 53″ + 67° 03′ 36″ = 135° 05′ 29″

Sch.—We might have omitted the computation of angle ABD in the first part, and DBC in the second, and have found instead the entire angle B from $\sin a : \sin b :: \sin A : \sin B$. To compute this requires the looking out of but two logarithms, since sin *a* is given in the second part (*c*), and sin A in the first part (*a*).

2. Given *a* = 97° 35′, *b* = 27° 08′ 22″, and A = 40° 51′ 18″, to project and compute the triangle. Between what limits must the

value of a be assigned in order that there may be two solutions? Between what limiting values of a is a solution impossible?

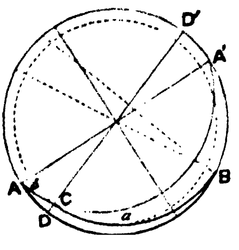


Fig. 65.

PROJECTION. See *Fig.* 65.

TRIGONOMETRICAL SOLUTION.—To find p, $\sin p = \sin b \sin \mathsf{A}$.

$\log \sin 27° 08′ 22″ = 9.659115$
$+ \log \sin 40° 51′ 18″ = \underline{9.815675}$
Rejecting 10 $= 9.474790 = \log \sin p$.
$\therefore p = 17° 21′ 40″$. [Reason for the species.]

We now observe that there is *one* and *only one* solution, since the arc a (97° 35′) cannot lie between **CD** (17° 21′ 40″) and b (27° 08′ 22″), but *can* lie between **CD** and **CA′** (180° − b = 152° 51′ 38″).

To find **AD**, $\cos \mathsf{A} = \cot b \tan \mathsf{AD}$ or $\tan \mathsf{AD} = \dfrac{\cos \mathsf{A}}{\cot b}$.

$\log \cos 40° 51′ 18″ = 9.878733$
$- \log \cot 27° 08′ 22″ = \underline{10.290226}$
Adding 10 $= 9.588507 = \log \tan \mathsf{AD}$. $\therefore$ **AD** $= 21° 11′ 30″.6$.

To find angle **ACD**, $\cos b = \cot \mathsf{A} \cot \mathsf{ACD}$, or $\cot \mathsf{ACD} = \dfrac{\cos b}{\cot \mathsf{A}}$.

$\log \cos 27° 08′ 22″ = 9.949340$
$- \log \cot 40° 51′ 18″ = 10.0630575$
Adding 10 $= 9.8862825 = \log \cot \mathsf{ACD}$. $\therefore$ **ACD** $= 52° 25′ 01″$.
[Reason for the species.]

To find **B**, $\sin p = \sin a \sin \mathsf{B}$, or $\sin \mathsf{B} = \dfrac{\sin p}{\sin a}$.

$\log \sin p$ (as above) $= 9.474790$
$- \log \sin 97° 35′ = 9.996185$
Adding 10 $= 9.478605 = \log \sin \mathsf{B}$. $\therefore$ **B** $= 17° 31′ 09″$.
[Reason for the species.]

To find **DB**, $\cos a = \cos p \cos \mathsf{DB}$, or $\cos \mathsf{DB} = \dfrac{\cos a}{\cos p}$.

$\log \cos 97° 35′ = 9.120469$
$- \log \cos 17° 21′ 40″ = \underline{9.979750}$
Adding 10 $= 9.140719 = \log \cos \mathsf{DB}$. $\therefore$ **DB** $= 97° 56′ 51″.3$.
[Reason for the species.]

AB = **AD** + **DB** $= 21° 11′ 30″.6 + 97° 56′ 51″.3 = 119° 08′ 21″.9$.

To find **DCB**, $\sin p = \tan \mathsf{DB} \cot \mathsf{DCB}$, or $\cot \mathsf{DCB} =$

$\log \sin p$ (as above) $= 9.474790$
$- \log \tan 97° 56′ 51″.3 = \underline{10.855090}$
Adding 10 $= 8.619700 = \log \cot \mathsf{DCB}$. $\therefore$ **DCB** $= 92° 23′ 7″.7$.
[Reason for the species.]

ACB = **C** = **ACD** + **DCB** $= 52° 25′ 01″ + 92° 23′ 7″.7 = 144° 48′ 8″.7$.

Finally, we observe that, if any value were assigned to a between b (27° 08′ 22″) and **CD** (17° 21′ 40″) there would be *two* solutions; since for such values the side a could lie on both sides of **CD**. But, for any value of a less than **CD** (17° 21′ 40″), there would be *no* solution; since **CD** is the shortest distance from **C** to the arc **ABA′**. Also, for any value of a greater than the arc **CA** (152° 51′ 38″), there would be *no* solution, as such an arc would fall between **CA** and **CD′** (if not > **CD′**), and consequently would make $c > 180°$.

SCH.—Such examples as this and the preceding can be more expeditiously solved by using p in each equation in solving the triangles **ACD** and **DCB**. By this means and using (*135*) to determine the side c, the solution can be effected with only 12 logarithms. Thus in Ex. 2.

1st.	To find p,	$\sin p = \sin b \sin$ **A**,	requires 3	logarithms	
2d.	To find **ACD**,	$\cos$ **ACD** $= \cot b \tan p$,	requires 3	"	
3d.	To find **DCB**,	$\cos$ **DCB** $= \cot a \tan p$,	requires 2	"	(log tan p being known).
4th.	To find **B**,	$\sin p = \sin a \sin$ **B**,	requires 2		(log sin p being known).
5th.	To find c,	$\sin$ **A** : $\sin$ **C** :: $\sin a : \sin c$,	requires 2	"	(log's of sin **A** …ng kn…

3. Given $a = 76° 35′ 36″$, $b = 50° 10′ 30″$, project and solve the triangle.

PROJECTION.—See *Fig.* 66.

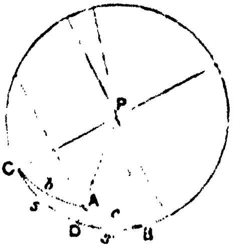


FIG. 66

TRIGONOMETRICAL SOLUTION.—1st. To find the segments **CD** and **DB**, we have,

$$\tan\tfrac{1}{2}(s + s'),$$

or

$$\tan\tfrac{1}{2}a : \tan\tfrac{1}{2}(b + c) :: \tan\tfrac{1}{2}(b - c) : \tan\tfrac{1}{2}(s - s').$$

Computing by logarithms.

a. c. $\log\tan\frac{1}{2}a$ = log tan 38° 17′ 48″

+ $\log\tan\frac{1}{2}(b + c)$ = log tan 45° 05′ 20″

+ $\log\tan\frac{1}{2}(b - c)$ = log tan 5° 05′ 10″

Rejecting 10 = 9.053314 = log tan ½

) =

In order to determine whether s or s' is the greater,* we observe the signs of the proportion, and finding $\tan\frac{1}{2}(s - s')$ positive, know that $s > s'$.

Hence, $s = \frac{1}{2}(s + s') + \frac{1}{2}(s - s') = 38° 17′ 48″ + 6° 27′ 02″ = 44° 44′ 50″$,

and $s' = \frac{1}{2}(s + s') - \frac{1}{2}(s - s') = 38° 17′ 48″ - 6° 27′ 02″ = 31° 50′ 46″$.

The angles sought are now readily found by computing the two right angled triangles **ADC** and **ADB**.

…ermines such facts as this, the species of parts, and the number of solutions in ambiguous cases, *the student should not rely upon it*, but determine …ch fact upon purely trigonometrical considerations, merely using the …
…s to the …

Or, having computed C from the triangle ACD, we may find A and B more expeditiously by using the proportions,

$$\sin c : \sin b :: \sin C : \sin B,$$

and

$$\sin c : \sin a :: \sin C : \sin A.$$

The angles are $C = 34^\circ\ 15'\ 03''$, $A = 121^\circ\ 36'\ 12''$, and $B = 42^\circ\ 15'\ 18''$.

4. Given $A = 128^\circ\ 45'$, $C = 30^\circ\ 35'$, and $a = 68^\circ\ 50'$, to solve the triangle. What values of A give two solutions? What none?

$c = 37^\circ\ 28'\ 20''$, $b = 40^\circ\ 09'\ 04''$, and $B = 32^\circ\ 37'\ 58''$.

5. Given $A = 129^\circ\ 05'\ 28''$, $B = 142^\circ\ 12'\ 42''$, and $C = 105^\circ\ 08'\ 10''$, to solve the triangle.

$a = 135^\circ\ 49'\ 20''$, $b = 146^\circ\ 37'\ 15''$, and $c = 60^\circ\ 04'\ 54''$.

6. Given $a = 68^\circ\ 46'\ 02''$, $b = 37^\circ\ 10'$, and $C = 43^\circ\ 37'\ 38''$, to project and solve the triangle.

$A = 116^\circ\ 22'\ 22''$, $B = 35^\circ\ 29'\ 54''$, and $c = 45^\circ\ 52'\ 34''$.

7. Given $a = 40^\circ\ 16'$, $b = 47^\circ\ 44'$, and $A = 52^\circ\ 34'$, to project and solve the triangle. What values of a give but one solution? What none?

There are two triangles.—In the 1st, $c = 53^\circ\ 19'\ 20''$, $B = 65^\circ\ 16'\ 35''$, and $C = 79^\circ\ 52'\ 21''$. In the 2d, $c = 14^\circ\ 18'\ 22''$, $B = 114^\circ\ 43'\ 25''$, and $C = 17^\circ\ 39'\ 22''$.

8. Given $a = 62^\circ\ 38'$, $b = 10^\circ\ 13'\ 19''$, and $C = 150^\circ\ 24'\ 12''$, to project and solve the triangle.

$A = 27^\circ\ 31'\ 44''$, $B = 5^\circ\ 17'\ 58''$, and $c = 71^\circ\ 37'\ 06''$.

9. Given $a = 56^\circ\ 40'$, $b = 83^\circ\ 13'$, and $c = 114^\circ\ 30'$, to project and solve the triangle.

$A = 48^\circ\ 31'\ 18''$, $B = 62^\circ\ 55'\ 44''$, and $C = 125^\circ\ 18''\ 56''$.

10. Given $A = 50^\circ\ 12'$, $B = 58^\circ\ 08'$, and $a = 62^\circ\ 42'$, to solve the triangle. What values of A give but one solution? What none?

There are two solutions.—1st, $b = 79^\circ\ 12'\ 10''$, $c = 119^\circ\ 03'\ 26''$, and $C = 130^\circ\ 54'\ 28''$. 2d, $b = 100^\circ\ 47'\ 50''$, $c = 152^\circ\ 14'\ 18''$, and $C = 156^\circ\ 15'\ 06''$.

11. Given $A = 36^\circ\ 25'$, $B = 42^\circ\ 17'\ 10''$, and $C = 95^\circ\ 10'\ 05''$, to project and solve the triangle.

12. Given $a = 124^\circ\ 53'$, $b = 31^\circ\ 19'$, and $c = 171^\circ\ 48'\ 22''$, to solve the triangle.

13. Given $a = 150^\circ\ 17'\ 23''$, $b = 43^\circ\ 12'$, and $c = 82^\circ\ 50'\ 12''$, to solve the triangle.

14. Given $a = 115^\circ\ 20'\ 10''$, $b = 57^\circ\ 30'\ 06''$, and $A = 126^\circ\ 37'\ 30''$, to solve the triangle.

15. Given $A = 109^\circ\ 55'\ 42''$, $B = 116^\circ\ 38'\ 33''$, and $C = 120^\circ\ 43'\ 37''$, to solve the triangle.

16. Given $A = 50^\circ$, $b = 60^\circ$, and $a = 40^\circ$, to solve the triangle.

17. Given $a = 50^\circ\ 45'\ 20''$, $b = 69^\circ\ 12'\ 40''$, and $A = 44^\circ\ 22'\ 10''$, to project and solve the triangle.

There are two solutions.—1st, $B = 57^\circ\ 34'\ 51''$, $C = 115^\circ\ 57'\ 51''$, and $c = 95^\circ\ 18'\ 16''$. 2d, $B = 122^\circ\ 25'\ 09''$, $C = 25^\circ\ 44'\ 32''$, and $c = 28^\circ\ 45'\ 05''$.

18. Given $b = 99^\circ\ 40'\ 48''$, $c = 100^\circ\ 49'\ 30''$, and $A = 65^\circ\ 33'\ 10''$, to project and solve the triangle.

$a = 64^\circ\ 23'\ 15''$, $B = 95^\circ\ 38'\ 04''$, and $C = 97^\circ\ 26'\ 29''$.

19. Given $A = 48^\circ\ 30'$, $B = 125^\circ\ 20'$, and $C = 62^\circ\ 54'$, to solve the triangle.

$a = 56^\circ\ 39'\ 30''$, $b = 114^\circ\ 29'\ 58''$, and $c = 83^\circ\ 12'\ 06''$.

20. Given $C = 54^\circ\ 15'\ 03''$, $B = 40^\circ\ 18'\ 13''$, and $a = 70^\circ\ 30'\ 30''$, to solve the triangle.

21. Given $A = 47^\circ\ 54'\ 21''$, $c = 61^\circ\ 04'\ 56''$, and $a = 40^\circ\ 31'\ 20''$, to project and solve the triangle.

22. Given $A = 50^\circ\ 10'\ 10''$, $b = 69^\circ\ 34'\ 35''$, and $a = 120^\circ\ 30'\ 30''$, to project and solve the triangle.

SECTION III.

GENERAL FORMULÆ.

[NOTE.—This section is designed for such as make mathematics a specialty. The preceding sections are thought sufficient for the general student.]

143. Prop.—*In a Spherical Triangle the cosine of any side is equal to the product of the cosines of the other two sides, plus the product of the sines of those sides into the cosine of their included angle; that is,*

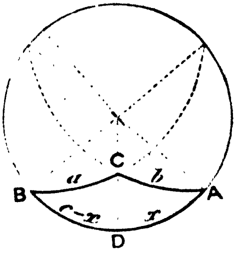


FIG. 67.

$$\left.\begin{aligned}(1)\ \cos a &= \cos b\cos c + \sin b\sin c\cos A;\\(2)\ \cos b &= \cos a\cos c + \sin a\sin c\cos B;\\(3)\ \cos c &= \cos a\cos b + \sin a\sin b\cos C.\end{aligned}\right\}\ \text{A.}$$

DEM.—From *Fig.* 67, we have,

$$\cos a = \cos(c-x)\cos p$$

$$= \frac{\cos(c-x)\cos b}{\cos x}\quad\left[\text{since }\cos p = \frac{\cos b}{\cos x}\right]$$

$$= \frac{\cos b\cos c\cos x + \cos b\sin c\sin x}{\cos x}\quad[\text{expanding }\cos(c-x)]$$

$$= \cos b\cos c + \cos b\sin c\tan x\quad\left[\text{since }\frac{\sin x}{\cos x} = \tan x\right]$$

$$= \cos b\cos c + \sin b\sin c\cos A\quad\left[\text{since }\cos A = \cot b\tan x = \frac{\cos b\tan x}{\sin b}\right].$$

In a similar manner (2) and (3) may be produced.

144. COR. 1.—*From set* A, *by passing to the polar triangle, we have,*

$$\left.\begin{aligned}(1)\ \cos A &= -\cos B\ \cos C + \sin B\ \sin C\ \cos a;\\(2)\ \cos B &= -\cos A\ \cos C + \sin A\ \sin C\ \cos b;\\(3)\ \cos C &= -\cos A\ \cos B + \sin A\ \sin B\ \cos c.\end{aligned}\right\}\ \text{B.}$$

DEM.—Letting a', b', c', A', B', and C' represent the parts of the polar triangle, we have $a = 180° - A'$, $b = 180° - B'$, $c = 180° - C'$, $A = 180° - a'$, $B = 180° - b'$, and $C = 180° - c'$. Whence, substituting in (1) A, we have,

$$\cos(180° - A') = \cos(180° - B')\cos(180° - C') \\ + \sin(180° - B')\sin(180° - C')\cos(180° - a'),$$

or, $$\cos A' = -\cos B' \cos C' + \sin B' \sin C' \cos a',$$

since $\cos(180° - A') = -\cos A'$, etc.; and $\sin(180° - B') = \sin B'$, etc.

Finally, dropping the accents, since the results are general, and treating (2) and (3) of set A in the same way, we have set B.

145. Cor. 2.—*From* A *and* B *we readily find the angles in terms of the sides, and the sides in terms of the angles. Thus, from* A,

$$\left.\begin{aligned} &(1)\ \cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}; \\ &(2)\ \cos B = \frac{\cos b - \cos a \cos c}{\sin a \sin c}; \\ &(3)\ \cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}. \end{aligned}\right\} \text{A}'.$$

From B,

$$\left.\begin{aligned} &(1)\ \cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}; \\ &(2)\ \cos b = \frac{\cos B + \cos A \cos C}{\sin A \sin C}; \\ &(3)\ \cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B}. \end{aligned}\right\} \text{B}'.$$

146. *Prop.*—*Formulæ* A', *and* B', *adapted to logarithmic computation, become,*

$$\left.\begin{aligned} &(1)\ \sin \tfrac{1}{2} A = \sqrt{\frac{\sin(\frac{1}{2}s - b)\sin(\frac{1}{2}s - c)}{\sin b \sin c}}; \\ &(2)\ \sin \tfrac{1}{2} B = \sqrt{\frac{\sin(\frac{1}{2}s - a)\sin(\frac{1}{2}s - c)}{\sin a \sin c}}; \\ &(3)\ \sin \tfrac{1}{2} C = \sqrt{\frac{\sin(\frac{1}{2}s - a)\sin(\frac{1}{2}s - b)}{\sin a \sin b}}. \end{aligned}\right\} \text{A}''.$$

And
$$\left.\begin{aligned}
(1)\ \sin\tfrac{1}{2}a &= \sqrt{-\frac{\cos\frac{1}{2}S\cos(\frac{1}{2}S - A)}{\sin B\sin C}};\\
(2)\ \sin\tfrac{1}{2}b &= \sqrt{-\frac{\cos\frac{1}{2}S\cos(\frac{1}{2}S - B)}{\sin A\sin C}};\\
(3)\ \sin\tfrac{1}{2}c &= \sqrt{-\frac{\cos\frac{1}{2}S\cos(\frac{1}{2}S - C)}{\sin A\sin B}}.
\end{aligned}\right\}\ B''.$$

Dem.—Subtracting each member of (1) A′ from 1, we have,

$$1 - \cos A = 1 - \frac{\cos a - \cos b\cos c}{\sin b\sin c} = \frac{\cos b\cos c + \sin b\sin c - \cos a}{\sin b\sin c}.$$

$$\therefore\ 2\sin^2\tfrac{1}{2}A = \frac{\cos(b - c) - \cos a}{\sin b\sin c},\ \text{since } 1 - \cos A = 2\sin^2\tfrac{1}{2}A\ (\textit{62, 5}),$$

and $\cos b\cos c + \sin b\sin c = \cos(b - c)$ (*55*, D).

Now letting $y = b - c$, and $x = a$, we see from (*59*, D′) that $\cos(b - c) - \cos a = 2\sin\frac{1}{2}(a + b - c)\sin\frac{1}{2}(a + c - b)$.

Hence,
$$2\sin^2\tfrac{1}{2}A = \frac{2\sin\frac{1}{2}(a + b - c)\sin\frac{1}{2}(a + c - b)}{\sin b\sin c},$$

or,
$$\sin\tfrac{1}{2}A = \sqrt{\frac{\sin\frac{1}{2}(a + b - c)\sin\frac{1}{2}(a + c - b)}{\sin b\sin c}}.$$

Finally, putting $s = a + b + c$, whence $\frac{1}{2}(a + b - c) = \frac{1}{2}s - c$,

and
$$\tfrac{1}{2}(a + c - b) = \tfrac{1}{2}s - b,$$

we have,
$$\sin\tfrac{1}{2}A = \sqrt{\frac{\sin(\frac{1}{2}s - b)\sin(\frac{1}{2}s - c)}{\sin b\sin c}}.$$

In like manner, (2) and (3) of set A′ reduce to (2) and (3) of set A″.

Again, subtracting each member of (1) set B′, from 1, we have, 9

$$1 - \cos a = 1 - \frac{\cos A + \cos B\cos C}{\sin B\sin C} = \frac{\sin B\sin C - \cos B\cos C - \cos A}{\sin B\sin C}.$$

$$\therefore\ 2\sin^2\tfrac{1}{2}a = \frac{-\cos(B + C) - \cos A}{\sin B\sin C} = -\frac{\cos(B + C) + \cos A}{\sin B\sin C}$$

$$= -\frac{2\cos\frac{1}{2}(A + B + C)\cos\frac{1}{2}(B + C + A)}{\sin B\sin C}.$$

Now putting $\mathsf{S} = \mathsf{A} + \mathsf{B} + \mathsf{C}$, whence $\frac{1}{2}(\mathsf{B} + \mathsf{C} - \mathsf{A}) = \frac{1}{2}\mathsf{S} - \mathsf{A}$, we have,

$$\sin \tfrac{1}{2}a = \sqrt{-\frac{\cos \frac{1}{2}\mathsf{S} \cos (\frac{1}{2}\mathsf{S} - \mathsf{A})}{\sin \mathsf{B} \sin \mathsf{C}}}.$$

In the same manner, (2) and (3) of B″ are deduced from (2) and (3) of B.

147. COR. 1.—*Passing to the polar triangle, A″ and B″ become*

$$\left.\begin{array}{ll}
(1) & \cos \frac{1}{2}a = \sqrt{\dfrac{\cos (\frac{1}{2}\mathsf{S} - \mathsf{B}) \cos (\frac{1}{2}\mathsf{S} - \mathsf{C})}{\sin \mathsf{B} \sin \mathsf{C}}}; \\[2ex]
(2) & \cos \frac{1}{2}b = \sqrt{\dfrac{\cos (\frac{1}{2}\mathsf{S} - \mathsf{A}) \cos (\frac{1}{2}\mathsf{S} - \mathsf{C})}{\sin \mathsf{A} \sin \mathsf{C}}}; \\[2ex]
(3) & \cos \frac{1}{2}c = \sqrt{\dfrac{\cos (\frac{1}{2}\mathsf{S} - \mathsf{A}) \cos (\frac{1}{2}\mathsf{S} - \mathsf{B})}{\sin \mathsf{A} \sin \mathsf{B}}}.
\end{array}\right\} \text{B}'''$$

And

$$\left.\begin{array}{ll}
(1) & \cos \frac{1}{2}\mathsf{A} = \sqrt{\dfrac{\sin \frac{1}{2}s \sin (\frac{1}{2}s - a)}{\sin b \sin c}}; \\[2ex]
(2) & \cos \frac{1}{2}\mathsf{B} = \sqrt{\dfrac{\sin \frac{1}{2}s \sin (\frac{1}{2}s - b)}{\sin a \sin c}}; \\[2ex]
(3) & \cos \frac{1}{2}\mathsf{C} = \sqrt{\dfrac{\sin \frac{1}{2}s \sin (\frac{1}{2}s - c)}{\sin a \sin b}}.
\end{array}\right\} \text{A}'''$$

148. SCH.—Formulæ A‴ and B‴ can be obtained directly from A and B, in a manner altogether similar to that in which A″ and B″ were deduced, by *adding* each member of the equations in sets A and B to 1, instead of subtracting, and observing that $1 + \cos x = 2\cos^2 \frac{1}{2}x$.

149. COR. 2.—*Dividing the formulæ of set A″ by the corresponding ones of set A‴; and, in a similar manner, those of B‴ by those of B″, and putting* $\sqrt{\dfrac{\sin (\frac{1}{2}s - a) \sin (\frac{1}{2}s - b) \sin (\frac{1}{2}s - c)}{\sin \frac{1}{2}s}} = k$,

and $\sqrt{\dfrac{\cos (\frac{1}{2}\mathsf{S} - \mathsf{A}) \cos (\frac{1}{2}\mathsf{S} - \mathsf{B}) \cos (\frac{1}{2}\mathsf{S} - \mathsf{C})}{-\cos \frac{1}{2}\mathsf{S}}} = \mathsf{K}$, *we have,*

$$\left.\begin{array}{ll}
(1) & \tan \frac{1}{2}\mathsf{A} = \sqrt{\dfrac{\sin (\frac{1}{2}s - b) \sin (\frac{1}{2}s - c)}{\sin \frac{1}{2}s \sin (\frac{1}{2}s - a)}} = \dfrac{k}{\sin (\frac{1}{2}s - a)}; \\[2ex]
(2) & \tan \frac{1}{2}\mathsf{B} = \sqrt{\dfrac{\sin (\frac{1}{2}s - a) \sin (\frac{1}{2}s - c)}{\sin \frac{1}{2}s \sin (\frac{1}{2}s - b)}} = \dfrac{k}{\sin (\frac{1}{2}s - b)}; \\[2ex]
(3) & \tan \frac{1}{2}\mathsf{C} = \sqrt{\dfrac{\sin (\frac{1}{2}s - a) \sin (\frac{1}{2}s - b)}{\sin \frac{1}{2}s \sin (\frac{1}{2}s - c)}} = \dfrac{k}{\sin (\frac{1}{2}s - c)}.
\end{array}\right\} \text{A}^{\text{iv}}$$

$$\left.\begin{array}{l}
(1)\ \cot \tfrac{1}{2}a = \sqrt{\dfrac{\cos(\frac{1}{2}S - B)\cos(\frac{1}{2}S - C)}{-\cos\frac{1}{2}S\cos(\frac{1}{2}S - A)}} = \dfrac{K}{\cos(\frac{1}{2}S - A)};\\
(2)\ \cot \tfrac{1}{2}b = \sqrt{\dfrac{\cos(\frac{1}{2}S - A)\cos(\frac{1}{2}S - C)}{-\cos\frac{1}{2}S\cos(\frac{1}{2}S - B)}} = \dfrac{K}{\cos(\frac{1}{2}S - B)};\\
(3)\ \cot \tfrac{1}{2}c = \sqrt{\dfrac{\cos(\frac{1}{2}S - A)\cos(\frac{1}{2}S - B)}{-\cos\frac{1}{2}S\cos(\frac{1}{2}S - C)}} = \dfrac{K}{\cos(\frac{1}{2}S - C)}.
\end{array}\right\} B'$$

SCH.—In these *formulæ* k is the tangent of the arc with which the inscribe circle is described, and K is the cotangent of the arc with which the circumscribed circle is described. Thus, using the common notation, we have t *Fig.* 68, AD = AD′ = $\frac{1}{2}s - a$, and angle PAD = $\frac{1}{2}$A; whence

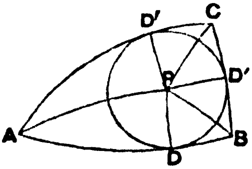


FIG. 68.

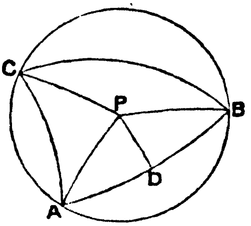


FIG. 69.

$$\sin AD = \cot PAD \times \tan PD = \frac{\tan PD}{\tan PAD},$$

or $$\tan \tfrac{1}{2}A = \frac{\tan PD}{\sin(\frac{1}{2}s - a)} = \frac{k}{\sin(\frac{1}{2}s - a)},\ [(1)\ A^{iv}].\quad \therefore k = \tan PD.$$

From *Fig.* 69, we have, AD = $\frac{1}{2}c$, and angle PAD = $\frac{1}{2}$S − C.

Hence, $$\cos(\tfrac{1}{2}S - C) = \cot AP \times \tan\tfrac{1}{2}c, \text{ or } \tan\tfrac{1}{2}c = \frac{\cos(\frac{1}{2}S - C)}{\cot AP},$$

or $$\cot\tfrac{1}{2}c = \frac{\cot AP}{\cos(\frac{1}{2}S - C)} = \frac{K}{\cos(\frac{1}{2}S - C)},\ [(3)\ B^{iv}].\quad \therefore K = \cot AP.$$

GAUSS'S EQUATIONS.

150. Prob.—*To deduce Gauss's Equations, which are*

$$(1)\ \frac{\sin\frac{1}{2}(A + B)}{\cos\frac{1}{2}C} = \frac{\cos\frac{1}{2}(a - b)}{\cos\frac{1}{2}c};$$

$$(2)\ \frac{\sin\frac{1}{2}(A - B)}{\cos\frac{1}{2}C} = \frac{\sin\frac{1}{2}(a - b)}{\sin\frac{1}{2}c};$$

$$(3)\ \frac{\cos\frac{1}{2}(A+B)}{\sin\frac{1}{2}C}=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c};$$

$$(4)\ \frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}=\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}.$$

SOLUTION.—From A, page 25, we have,

$$\sin(\tfrac{1}{2}A+\tfrac{1}{2}B), \text{ or } \sin\tfrac{1}{2}(A+B)=\sin\tfrac{1}{2}A\cos\tfrac{1}{2}B+\cos\tfrac{1}{2}A\sin\tfrac{1}{2}B.$$

bstituting in the second member the values of $\sin\frac{1}{2}A$, $\cos\frac{1}{2}B$, $\cos\frac{1}{2}A$, and n $\frac{1}{2}B$, from A″ and A‴, there results,

$$\text{in}\ \tfrac{1}{2}(A+B)=\frac{\sin(\frac{1}{2}s-b)}{\sin c}\sqrt{\frac{\sin\frac{1}{2}s\sin(\frac{1}{2}s-c)}{\sin a\sin b}}+\frac{\sin(\frac{1}{2}s-a)}{\sin c}\sqrt{\frac{\sin\frac{1}{2}s\sin(\frac{1}{2}s-c)}{\sin a\sin b}}$$

$$=\frac{\sin(\frac{1}{2}s-b)+\sin(\frac{1}{2}s-a)}{\sin c}\sqrt{\frac{\sin\frac{1}{2}s\sin(\frac{1}{2}s-c)}{\sin a\sin b}}.\quad [\text{See (3) A‴}].$$

$$=\frac{\sin(\frac{1}{2}s-b)+\sin(\frac{1}{2}s-a)}{\sin c}\cos\tfrac{1}{2}C.$$

But $\sin(\frac{1}{2}s-b)+\sin(\frac{1}{2}s-a)=\sin(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}c-b)+\sin(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}c-a)$

$$=\sin[\tfrac{1}{2}c+\tfrac{1}{2}(a-b)]+\sin[\tfrac{1}{2}c-\tfrac{1}{2}(a-b)]$$

$$=2\sin\tfrac{1}{2}c\cos\tfrac{1}{2}(a-b).\quad (\textit{59}, A).$$

Also, $\sin c=2\sin\frac{1}{2}c\cos\frac{1}{2}c$.

Substituting these values, the preceding becomes,

$$\sin\tfrac{1}{2}(A+B)=\frac{2\sin\frac{1}{2}c\cos\frac{1}{2}(a-b)}{2\sin\frac{1}{2}c\cos\frac{1}{2}c}\cos\tfrac{1}{2}C.$$

$$\therefore\ \frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}C}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}.\quad (1)$$

In like manner starting with

$$\sin(\tfrac{1}{2}A-\tfrac{1}{2}B), \text{ or } \sin\tfrac{1}{2}(A-B)=\sin\tfrac{1}{2}A\cos\tfrac{1}{2}B-\cos\tfrac{1}{2}A\sin\tfrac{1}{2}B,$$

there results, $$\frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c}.\quad (2)$$

Starting with $\cos(\frac{1}{2}A+\frac{1}{2}B)$, or $\cos\frac{1}{2}(A+B)=\cos\frac{1}{2}A\cos\frac{1}{2}B-\sin\frac{1}{2}A\sin\frac{1}{2}B$,

there results, $$\frac{\cos\frac{1}{2}(A+B)}{\sin\frac{1}{2}C}=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}.\quad (3)$$

Starting with $\cos(\frac{1}{2}A-\frac{1}{2}B)$, or $\cos\frac{1}{2}(A-B)=\cos\frac{1}{2}A\cos\frac{1}{2}B+\sin\frac{1}{2}A\sin\frac{1}{2}B$,

there results, $$\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}=\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}.\quad (4)$$

NAPIER'S ANALOGIES.

***151. Prob.**—To deduce Napier's Analogies, which are*

$$(1)\quad \frac{\tan\frac{1}{2}(A+B)}{\cot\frac{1}{2}C}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)};$$

$$(2)\quad \frac{\tan\frac{1}{2}(A-B)}{\cot\frac{1}{2}C}=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)};$$

$$(3)\quad \frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}c}=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)};$$

$$(4)\quad \frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}c}=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}.$$

SOLUTION.—To deduce (1), divide the 1st of Gauss's Equations by the 3d. To deduce (2), divide the 2d of Gauss's by the 4th. To deduce (3), divide the 4th of Gauss's by the 3d. To deduce (4), divide the 2d of Gauss's by the 1st.

152. SCH.—In using these formulæ the species must be carefully attended to. Thus in (1), $\cot\frac{1}{2}C$ and $\cos\frac{1}{2}(a-b)$ are necessarily $+$; hence $\tan\frac{1}{2}(A+B)$ and $\cos\frac{1}{2}(a+b)$ are of the same sign with each other. In (2), $\cot\frac{1}{2}C$ and $\sin\frac{1}{2}(a+b)$ are both $+$; hence, $\tan\frac{1}{2}(A-B)$ and $\sin\frac{1}{2}(a-b)$ are of the same sign with each other. And similar inspections may be made upon (3) and (4).

EXERCISES.

153. The proposition that "The sines of the angles are to each other as the sines of their opposite sides" (***135***), Napier's Analogies (***151***), and formulæ A^{iv}, B^{iv} (***149***) are sufficient, in themselves, to effect the solution of all cases of oblique spherical triangles; and for practical purposes they generally require less labor than Napier's Rules. We give a few solutions and refer the student to the preceding Exercises for further practice.

1. Given $a = 100°$; $C = 5°$ and $b = 10°$, to solve the triangle. (*Prob.* 1, Case 1st, ***137.***)

1st. To find A and B we have,

$$\cos\tfrac{1}{2}(a+b) : \cos\tfrac{1}{2}(a-b) :: \cot\tfrac{1}{2}C : \tan\tfrac{1}{2}(A+B);$$

and $$\sin\tfrac{1}{2}(a+b) : \sin\tfrac{1}{2}(a-b) :: \cot\tfrac{1}{2}C : \tan\tfrac{1}{2}(A-B)\quad [\textbf{150}\ (1)\ (2)]$$

Computing by logarithms, we have,

$$\begin{aligned}
\text{ar. co. log cos } [\tfrac{1}{2}(a+b) = 55°] &= 0.241409 \\
+ \text{ log cos } [\tfrac{1}{2}(a-b) = 45°] &= 9.849485 \\
+ \text{ log cot } [\tfrac{1}{2}\text{C} = 2°\ 30'] &= \underline{11.359907} \\
\text{Rejecting 10 } &= 11.450801 = \text{log tan } \tfrac{1}{2}(\text{A} + \text{B}). \\
&\therefore\ \tfrac{1}{2}(\text{A} + \text{B}) = 87°\ 58'\ 18''.
\end{aligned}$$

$$\begin{aligned}
\text{ar. co. log sin } [\tfrac{1}{2}(a+b) = 55°] &= 0.086635 \\
+ \text{ log sin } [\tfrac{1}{2}(a-b) = 45°] &= 9.849485 \\
+ \text{ log cot } [\tfrac{1}{2}\text{C} = 2°\ 30'] &= \underline{11.359907} \\
\text{Rejecting 10 } &= 11.296027 = \text{log tan } \tfrac{1}{2}(\text{A} - \text{B}). \\
&\therefore\ \tfrac{1}{2}(\text{A} - \text{B}) = 87°\ 06'\ 16''
\end{aligned}$$

The signs of all the terms being +, $\frac{1}{2}(A + B)$ and $\frac{1}{2}(A - B)$ are both less than 90°.

$$\tfrac{1}{2}(\text{A} + \text{B}) + \tfrac{1}{2}(\text{A} - \text{B}) = \text{A} = 87°\ 58'\ 18'' + 87°\ 06'\ 16'' = 175°\ 04'\ 34''$$

$$\tfrac{1}{2}(\text{A} + \text{B}) - \tfrac{1}{2}(\text{A} - \text{B}) = \text{B} = 87°\ 58'\ 18'' - 87°\ 06'\ 16'' = 0°\ 52'\ 02''.$$

2d. To find c. This may be found from the proportion,

$$\sin \text{A} : \sin \text{C} :: \sin a : \sin c,$$

or from the 3d or 4th of Napier's Analogies. We use the last, though the first s equally expeditious.

$$\sin \tfrac{1}{2}(\text{A} - \text{B}) : \sin \tfrac{1}{2}(\text{A} + \text{B}) :: \tan \tfrac{1}{2}(a - b) : \tan \tfrac{1}{2}c.$$

$$\begin{aligned}
\text{ar. co. log sin } [\tfrac{1}{2}(\text{A} - \text{B}) = 87°\ 06'\ 16''] &= 0.000555 \\
+ \text{ log sin } [\tfrac{1}{2}(\text{A} + \text{B}) = 87°\ 58'\ 18''] &= 9.999728 \\
+ \text{ log tan } [\tfrac{1}{2}(a - b) = 45°] &= \underline{10.000000} \\
\text{Rejecting 10 } &= 10.000283 = \text{log tan } \tfrac{1}{2}c. \\
&\therefore\ c = 90°\ 02'\ 14''.
\end{aligned}$$

2. Given $\text{A} = 135°\ 05'\ 28''.6$, $\text{C} = 50°\ 30'\ 08''.6$, and $b = 69°\ 34'\ 56''.2$, to solve the triangle.

1st. To find a and c. The 3d and 4th of Napier's Analogies give,

$$\cos \tfrac{1}{2}(\text{A} + \text{C}) : \cos \tfrac{1}{2}(\text{A} - \text{C}) : \tan \tfrac{1}{2}b : \tan \tfrac{1}{2}(a + c);$$

and

$$\sin \tfrac{1}{2}(\text{A} + \text{C}) : \sin \tfrac{1}{2}(\text{A} - \text{C}) :: \tan \tfrac{1}{2}b : \tan \tfrac{1}{2}(a - c).$$

Computing by logarithms, we have

$$\begin{aligned}
\text{ar. co. log cos } [\tfrac{1}{2}(\text{A} + \text{C}) = 92°\ 47'\ 48''.6] &= 1.3116286\ ^{*} \\
+ \text{ log cos } [\tfrac{1}{2}(\text{A} - \text{C}) = 42°\ 17'\ 40\ ] &= 9.8690535 \\
+ \text{ log tan } [\tfrac{1}{2}b = 34°\ 47'\ 28''.1] &= \underline{9.8418527} \\
\text{Rejecting 10 } &= 11.0225348 = \text{log tan } \tfrac{1}{2}(a + c). \\
&\therefore\ \tfrac{1}{2}(a + c) = 95°\ 25'\ 25''.
\end{aligned}$$

$\frac{1}{2}(a + c) > 90°$, since $\cos \frac{1}{2}(A + C)$ is −, $\cos \frac{1}{2}(A - C)$ is +, and $\tan \frac{1}{2}b$ is +, making $\tan \frac{1}{2}(a + c)$ −.

---

* These logarithms are taken from 7-place tables, in order to obtain the tenths of seconds accurately.

$$
\begin{aligned}
\text{ar. co. log sin } [\tfrac{1}{2}(A + C) = 92^\circ\ 47'\ 48''.6] &= 0.0005176\\
+ \text{ log sin } [\tfrac{1}{2}(A - C) = 42^\circ\ 17'\ 40''\ ] &= 9.8279768\\
+ \text{ log tan } [\tfrac{1}{2}b = 34^\circ\ 47'\ 28''.1] &= \underline{9.8418527}\\
\text{Rejecting 10} &= 9.6703471 = \text{log tan } \tfrac{1}{2}(a - c).\\
\therefore\ \tfrac{1}{2}(a - c) &= 25^\circ\ 05'\ 05''.
\end{aligned}
$$

$\frac{1}{2}(a - c) < 90^\circ$, since the signs of the terms are all +.

$\frac{1}{2}(a + c) + \frac{1}{2}(a - c) = a = 120^\circ\ 30'\ 30''$, and $\frac{1}{2}(a + c) - \frac{1}{2}(a - c) = c = 70^\circ\ 20'\ 20''$.

2d. To find B. Either of the 1st two of Napier's Analogies will give B. Thus (1) becomes,

$$\cos \tfrac{1}{2}(a - c) : \cos \tfrac{1}{2}(a + c) :: \tan \tfrac{1}{2}(A + C) : \cot \tfrac{1}{2}B;$$

and (2) $\sin \frac{1}{2}(a - c) : \sin \frac{1}{2}(a + c) :: \tan \frac{1}{2}(A - C) : \cot \frac{1}{2}B.$

But as $\frac{1}{2}(a + c)$ is so near 90°, it will be better to use the second of these than the first. Or we may with equal accuracy use,

$$\sin c : \sin b :: \sin C : \sin B.$$

$$
\begin{aligned}
\text{ar. co. log sin } (c = 70^\circ\ 20'\ 20''\ \ ) &= 0.0260878\\
+ \text{ log sin } (b = 69^\circ\ 34'\ 56''.2) &= 9.9718202\\
+ \text{ log sin } (C = 50^\circ\ 30'\ 08''.6) &= \underline{9.8874210}\\
\text{Rejecting 10} &= 9.8853290 = \text{log sin B}.\quad \therefore\ B = 50^\circ\ 10'\ 10''.
\end{aligned}
$$

3. Given $a = 50^\circ\ 45'\ 20''$, $b = 69^\circ\ 12'\ 40''$, and $A = 44^\circ\ 22'\ 10''$, to solve the triangle.

1st. To find B. $\sin a : \sin b :: \sin A : \sin B.$

$$
\begin{aligned}
\text{ar. co. log sin } (a = 50^\circ\ 45'\ 20'') &= 0.1110044\\
+ \text{ log sin } (b = 69^\circ\ 12'\ 40'') &= 9.9707626\\
+ \text{ log sin } (A = 44^\circ\ 22'\ 10'') &= \underline{9.8446525}\\
\text{Rejecting 10} &= 9.9264195 = \text{log sin B}.\quad \therefore\ B - 57^\circ\ 34'\ 51''.4,
\end{aligned}
$$

and 122° 25′ 08″.6. There are two solutions, since a is intermediate in value between p and both b and $180^\circ - b$.*

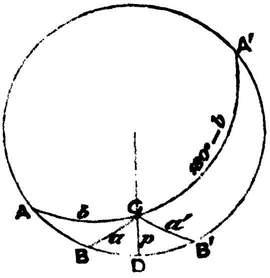


FIG. 70.

* The determination of the species of B, or what is the same thing, the number of solutions, can usually be effected by a simple inspection without any computation whatever. Thus, $\sin p = \sin b \sin A$, the logarithms of which are given above, as is log sin a. Now, as both a and p are $< 90^\circ$, and $\log \sin p < \log \sin a$, $p < a$. But $a < b$, and also less than $180^\circ - b$. All this can be seen at a glance.

To find C and c of the larger triangle in which $B = 57° 34' 51''.4$

Napier's 1st gives

$$\begin{aligned}
\text{ar. co. } \log\cos[\tfrac{1}{2}(b-a) &= 9° 13' 40''] = 0.0056570\\
+\ \log\cos[\tfrac{1}{2}(b+a) &= 59° 59'] = 9.6991887\\
+\ \log\tan[\tfrac{1}{2}(B+A) &= 50° 58' 30''.7] = 10.0912464\\
\text{Rejecting } 10 &= 9.7960921 = \log\cot\tfrac{1}{2}C.\\
\therefore C &= 115° 57' 50''.7.
\end{aligned}$$

Napier's 3d gives

$$\begin{aligned}
\text{ar. co. } \log\cos[\tfrac{1}{2}(B-A) &= 6° 36' 20''.7] = 0.0028928\\
+\ \log\cos[\tfrac{1}{2}(B+A) &= 50° 58' 30''.7] = 9.7991039\\
+\ \log\tan[\tfrac{1}{2}(b+a) &= 59° 59'] \quad 10.2382689\\
\text{Rejecting } 10 & \quad 10.0402656 = \log\tan\tfrac{1}{2}c.\\
\therefore c &= 95° 18' 16''.4,
\end{aligned}$$

3d. To find C and c of the smaller triangle in which $B = 122° 25' 08''.6$. Using the same *formulæ* as before.

$$\begin{aligned}
\text{ar. co. } \log\cos[\tfrac{1}{2}(b-a) &= 9° 13' 40''] \quad 0.0056570\\
+\ \log\cos[\tfrac{1}{2}(b+a) &= 59° 59'] \quad 9.6991887\\
+\ \log\tan[\tfrac{1}{2}(B+A) &= 83° 23' 39''.3] \quad 10.9362703\\
\text{Rejecting } 10 & \quad 10.6411160 = \log\cot\tfrac{1}{2}C.\\
C &= 25° 44' 31''.6.
\end{aligned}$$

$$\begin{aligned}
\text{ar. co. } \log\cos[\tfrac{1}{2}(B-A) &= 39° 01' 29''.3] = 0.1096506\\
+\ \log\cos[\tfrac{1}{2}(B+A) &= 83° 23' 39''.3] = 9.0608369\\
+\ \log\tan[\tfrac{1}{2}(b+a) &= 59° 59'] \quad 10.2382689\\
\text{Rejecting } 10 & \quad 9.4087564 = \log\tan\tfrac{1}{2}c.\\
\therefore c &= 28° 45' 05''.2.
\end{aligned}$$

154. SCH.—When Napier's Analogies are used for solving *Prob 2nd* (***130***), the most expeditious and elegant method of resolving the ambiguity, is by means of the analogies themselves. Thus, in the above example, after having found that $B = 57° 34' 51''.4$, or $122° 25' 08''.6$, or both, a simple inspection of the analogy next used will determine the number of solutions.

Napier's 1st may be written

$$\cot\tfrac{1}{2}C = \frac{\cos\tfrac{1}{2}(b+a)}{\cos\tfrac{1}{2}(b-a)}\tan\tfrac{1}{2}(B+A).$$

Now $\tfrac{1}{2}C < 90°$, hence $\cot\tfrac{1}{2}C$ is $+$. If, therefore, neither of the values of B renders $\cot\tfrac{1}{2}C$ $-$, there are *two* solutions. If one value renders $\cot\tfrac{1}{2}C$ $+$, and the other $-$, there is *one* solution and it corresponds to the value of B which makes $\cot\tfrac{1}{2}C$ $+$. If both values of B render $\cot\tfrac{1}{2}C$ $-$, there is *no* solution. In the last example, we see that $\cos[\tfrac{1}{2}(b+a) = 59° 59']$, and $\cos[\tfrac{1}{2}(b-a) = 9° 13' 40'']$ are both $+$. Also $\tan[\tfrac{1}{2}(B+A) = 50° 58' 30''.7$, or $83° 23' 39''.3$, or both] is $+$ for both values of B. Therefore there are *two* solutions.

4. Given $\mathsf{A} = 95^\circ\ 16'$, $\mathsf{B} = 80^\circ\ 42'\ 10''$, and $a = 57^\circ\ 38'$, to solve the triangle.

1st. To find b, $\sin \mathsf{A} : \sin \mathsf{B} :: \sin a : \sin b$.

$$\begin{aligned}
\text{ar. co. } \log \sin (\mathsf{A} &= 95^\circ\ 16') &&= 0.001837\\
+ \log \sin (\mathsf{B} &= 80^\circ\ 42'\ 10'') &&= 9.994257\\
+ \log \sin (a &= 57^\circ\ 38') &&= 9.926671\\
&\text{Rejecting } 10 &&= 9.922765 = \log \sin b.\\
&\therefore b = 56^\circ\ 49'\ 57'', \text{ or } 123^\circ\ 10'\ 03'', \text{ or both.}
\end{aligned}$$

2d. To find c, $\tan \frac{1}{2}c = \dfrac{\cos \frac{1}{2}(\mathsf{A} + \mathsf{B})}{\cos \frac{1}{2}(\mathsf{A} - \mathsf{B})} \tan \frac{1}{2}(a + b)$. Now for $b = 56^\circ\ 49'\ 57''$, $\tan \frac{1}{2}c$ is $+$; but for $b = 123^\circ\ 10'\ 03''$ $\tan \frac{1}{2}c$ is $-$; hence there is but one solution, and that corresponds to the smaller value of b.

$$\begin{aligned}
\text{ar. co. } \log \cos [\tfrac{1}{2}(\mathsf{A} - \mathsf{B}) &= 7^\circ\ 16'\ 55''] &&= 0.003517\\
+ \log \cos [\tfrac{1}{2}(\mathsf{A} + \mathsf{B}) &= 87^\circ\ 59'\ 05''] &&= 8.546124\\
+ \log \tan [\tfrac{1}{2}(a + b) &= 57^\circ\ 13'\ 58''] &&= 10.191352\\
&\text{Rejecting } 10 &&= 8.740993 = \log \tan \tfrac{1}{2}c.\\
&&&\therefore c = 6^\circ\ 18'\ 19''.
\end{aligned}$$

3d. To find C, we may use (1) or (2) of Napier's Analogies, or

$$\sin a : \sin c :: \sin \mathsf{A} : \sin \mathsf{C},$$

the last of which is the most expeditious.

$$\begin{aligned}
\text{ar. co. } \log \sin (a &= 57^\circ\ 38') &&= 0.073329\\
+ \log \sin (c &= 6^\circ\ 18'\ 19'') &&= 9.040705\\
+ \log \sin (\mathsf{A} &= 95^\circ\ 16') &&= 9.998163\\
&\text{Rejecting } 10 &&= 9.112197 = \log \sin \mathsf{C}. \therefore \mathsf{C} = 7^\circ\ 26'\ 22''
\end{aligned}$$

This value is taken for C instead of its supplement, since C is opposite the smallest side c.

5. Given $a = 70^\circ\ 14'\ 20''$, $b = 49^\circ\ 24'\ 10''$, and $c = 38^\circ\ 46'\ 10''$; to solve the triangle.

COMPUTATION.

$$\begin{aligned}
a &= 70^\circ\ 14'\ 20''\\
b &= 49^\circ\ 24'\ 10''\\
c &= 38^\circ\ 46'\ 10''\\
s &= 158^\circ\ 24'\ 40''\\
\tfrac{1}{2}s &= 79^\circ\ 12'\ 20'' &\quad \text{ar. co. } \log \sin &= 0.007753\\
\tfrac{1}{2}s - a &= 8^\circ\ 58'\ 00'' & \text{"} \quad \text{"} &= 9.192734\\
\tfrac{1}{2}s - b &= 29^\circ\ 48'\ 10'' & \text{"} \quad \text{"} &= 9.696370\\
\tfrac{1}{2}s - c &= 40^\circ\ 26'\ 10'' & \text{"} \quad \text{"} &= 9.811977\\
&& &\ 2)\underline{18.708834}\\
&& \therefore \log k &= 9.354417
\end{aligned}$$

$\log \tan \frac{1}{2}A = \log k - \log \sin (\frac{1}{2}s - a) + 10 = 10.161683. \quad \therefore A = 110° 51' 16''.$
$\log \tan \frac{1}{2}B = \log k - \log \sin (\frac{1}{2}s - b) + 10 = 9.658047. \quad \therefore B = 48° 56' 04''.$
$\log \tan \frac{1}{2}C = \log k - \log \sin (\frac{1}{2}s - c) + 10 = 9.542440. \quad \therefore C = 38° 26' 48''.$

6. Given $A = 109° 55' 42''$, $B = 116° 38' 33''$, and $C = 120° 43' 37''$, to solve the triangle.

COMPUTATION.

$A = 109° 55' 42''$
$B = 116° 38' 33''$
$C = 120° 43' 37''$
$S = 347° 17' 52''$
$\frac{1}{2}S = 173° 38' 56''$ ar. co. log cos = 0.003683
$\frac{1}{2}S - A = 63° 43' 14''$ " " 9.646158
$\frac{1}{2}S - B = 57° 00' 23''$ " " 9.736035
$\frac{1}{2}S - C = 52° 55' 19''$ " " 9.780247
2)19.165123
$\therefore$ log K 9.582561

$\log \cot \frac{1}{2}a = \log K - \log \cos (\frac{1}{2}S - A) + 10 = 9.936403. \quad \therefore a = 98° 21' 38''.$
$\log \cot \frac{1}{2}b = \log K - \log \cos (\frac{1}{2}S - B) + 10 = 9.846526. \quad \therefore b = 100° 50' 20''.$
$\log \cot \frac{1}{2}c = \log K - \log \cos (\frac{1}{2}S - C) + 10 = 9.802314. \quad \therefore c = 115° 13' 28''.$

SCH. 1.—The student can use the exercises in the preceding section to familiarize the methods here given. In doing so, it will be well for him to seek the most expeditious solutions. He will find that

Examples under PROB. 1 require 11 logarithms by Napier's Analogies and (*135*), and 12 logarithms by Napier's Rules and (*135*).

Examples under PROB. 2, when there is but one solution, require 10 logarithms by Napier's Analogies and (*135*), and 12 logarithms by Napier's Rules and (*135*). When there are two solutions, 15 logarithms are required by Napier's Analogies and (*135*), and only 14 by Napier's Rules alone, or by these rules and (*135*).

Examples under PROB. 3 require but 7 logarithms by the method given in this section and 13 by the previous method.

SCH. 2.—In cases in which the angles or sides are near the limits 0°, 90°, or 180°, so that the functions used in the particular solution change very rapidly in proportion to the arc, it is often possible to select one among the several methods which will give more accurate results than the others. There are also other methods which are better adapted to such cases than those here given. For these, as well as for much other interesting matter, and especially for the discussion of the General Spherical Triangle, American students have an excellent resource in the treatise of Professor Chauvenet of Washington University, St. Louis.

SECTION IV.

AREA OF SPHERICAL TRIANGLES.

155. Prob.*—Having the angles of a spherical triangle given, to find the area.*

Solution.—[The solution is given in Part II. (***613***), and we simply reproduce the result in order to give completeness to this section.] The area is equal to the ratio of the spherical excess to 90°, or $\frac{1}{2}\pi$, into the trirectangular triangle. That is, letting the sum of the angles be S°, the area K, and the radius of the sphere 1, whence the area of the trirectangular triangle is $\frac{1}{2}\pi$, we have

$$K = \frac{S - \pi}{\frac{1}{2}\pi} \times \frac{1}{2}\pi = S - \pi.$$

In the latter expression S is the sum of the angles in terms of the radius, *i. e.*, $S = \frac{S^\circ}{57^\circ.29578}$, or approximately, $S = \frac{S^\circ}{57^\circ.3}$ (***9***).

EXERCISES.

1. What is the area of a spherical triangle whose angles are 100°, 58°, and 62°, on a sphere whose diameter is 6 feet?

Solution. $K = S - \pi = \frac{220^\circ}{57^\circ.3} - 3.14159 = .698$, the area of a similar triangle on a sphere whose radius is 1. Hence, the area of the required triangle is $.698 \times 3^2 = 6.282$. [The method given in Part II. (***613***) is more expeditious, but it is our purpose to illustrate the form here given.]

2. What is the area of a spherical triangle whose angles are 170°, 135°, and 115°, on a sphere whose radius is 10 feet?

Ans. 418.875 square feet.

3. What is the area of a spherical triangle whose angles are 150°, 110°, and 60°, on a sphere whose radius is 3 feet?

---

156. Prob.*—Having the sides of a spherical triangle given, to find the area.*

Solution.—The angles may be found by (***148***), and then the area by (***155***).

But a more direct method is to find the spherical excess by means of *Lhuillier's* formula, which we will now produce.

$$\tfrac{1}{4}K = \tfrac{1}{4}(A + B + C - \pi)$$

Whence $\tan \tfrac{1}{4}K = \tan \tfrac{1}{4}[A + B + C - \pi] = \tan [\tfrac{1}{4}(A + B) - \tfrac{1}{4}(\pi - C)]$

$$= \frac{\sin \tfrac{1}{2}(A + B) - \sin \tfrac{1}{2}(\pi - C)}{\cos \tfrac{1}{2}(A + B) + \cos \tfrac{1}{2}(\pi - C)} \qquad \text{(7, page } \textit{31}\text{)}$$

$$= \frac{\sin \tfrac{1}{2}(A + B) - \cos \tfrac{1}{2}C}{\cos \tfrac{1}{2}(A + B) + \sin \tfrac{1}{2}C}$$

$$= \frac{[\cos \tfrac{1}{2}(a - b) - \cos \tfrac{1}{2}c] \cos \tfrac{1}{2}C}{[\cos \tfrac{1}{2}(a + b) + \cos \tfrac{1}{2}c] \sin \tfrac{1}{2}C} \qquad (\textit{150}, \text{1st and 3d})$$

$$= \frac{\cos \tfrac{1}{2}(a - b) - \cos \tfrac{1}{2}c}{\cos \tfrac{1}{2}(a + b) + \cos \tfrac{1}{2}c} \sqrt{\frac{\sin \tfrac{1}{2}s \sin (\tfrac{1}{2}s - c)}{\sin (\tfrac{1}{2}s - a) \sin (\tfrac{1}{2}s - b)}} \qquad (\textit{146}, \textit{147})$$

$$= \frac{\sin \tfrac{1}{4}(a + c - b) \sin \tfrac{1}{4}(b + c - a)}{\cos \tfrac{1}{4}(a + b + c) \cos \tfrac{1}{4}(a + b - c)} \sqrt{\frac{\sin \tfrac{1}{2}s \sin (\tfrac{1}{2}s - c)}{\sin (\tfrac{1}{2}s - a) \sin (\tfrac{1}{2}s - b)}} \qquad (\textit{59}, \text{C}', \text{D}')$$

$$= \sqrt{\frac{\sin^2 \tfrac{1}{2}(\tfrac{1}{2}s - b) \sin^2 \tfrac{1}{2}(\tfrac{1}{2}s - a) \sin \tfrac{1}{2}s \sin (\tfrac{1}{2}s - c)}{\cos^2 \tfrac{1}{4}s \cos^2 \tfrac{1}{2}(\tfrac{1}{2}s - c) \sin (\tfrac{1}{2}s - a) \sin (\tfrac{1}{2}s - b)}} \qquad (\text{as } s = a + b + c)$$

$$= \sqrt{\frac{\sin^2 \tfrac{1}{2}(\tfrac{1}{2}s - b) \sin^2 \tfrac{1}{2}(\tfrac{1}{2}s - a) \sin \tfrac{1}{4}s \cos \tfrac{1}{4}s \sin \tfrac{1}{2}(\tfrac{1}{2}s - c) \cos \tfrac{1}{2}(\tfrac{1}{2}s - c)}{\cos^2 \tfrac{1}{4}s \cos^2 \tfrac{1}{2}(\tfrac{1}{2}s - c) \sin \tfrac{1}{2}(\tfrac{1}{2}s - a) \cos \tfrac{1}{2}(\tfrac{1}{2}s - a) \sin \tfrac{1}{2}(\tfrac{1}{2}s - b) \cos \tfrac{1}{2}(\tfrac{1}{2}s - b)}} \qquad (\textit{56}).$$

$$\therefore \text{Tan } \tfrac{1}{4}K = \pm^{*} \sqrt{\tan \tfrac{1}{4}s \tan \tfrac{1}{2}(\tfrac{1}{2}s - a) \tan \tfrac{1}{2}(\tfrac{1}{2}s - b) \tan \tfrac{1}{2}(\tfrac{1}{2}s - c)} \qquad \text{(A)}$$

Having found K, the spherical excess, or what is the same thing, the area of a similar triangle on a sphere whose radius is 1, we have but to multiply K by the square of the radius in any given case.

EXERCISES.

1. Given $a = 98°$, $b = 110°$, and $c = 115°$, to find the area of a spherical triangle, on a sphere whose radius is 4000 miles.

COMPUTATION.

$\log \tan (\tfrac{1}{4}s = 80°\ 45') = 10.788185$

$+ \log \tan [\tfrac{1}{2}(\tfrac{1}{2}s - a) = 31°\ 45'] = 9.791563$

$+ \log \tan [\tfrac{1}{2}(\tfrac{1}{2}s - b) = 25°\ 45'] = 9.683356$

$+ \log \tan [\tfrac{1}{2}(\tfrac{1}{2}s - c) = 23°\ 15'] = 9.633098$

$2)\underline{39.896202}$

Rejecting $10 = 9.948101 = \log \tan \tfrac{1}{4}K$. $\therefore K = 166°\ 20'\ 20''$

---

* The + is always taken: otherwise, $\tfrac{1}{4}K$ being $> 90°$, K would be $> 360°$ which is impossible. (PART III., 256).

Whence area $= \frac{166° \; 20' \; 20''}{90°} \cdot \frac{1}{2}\pi (4000)^2 = \frac{166.34}{90} \cdot \frac{1}{2}\pi (4000)^2 = 46,450,440$, nearly.

2. Given $a = 70° \; 14' \; 20''$, $b = 49° \; 24' \; 10''$, and $c = 38° \; 46' \; 10''$, to find the area of a spherical triangle on a sphere whose diameter is 8 feet. *Ans.* 5.1, nearly.

157. Prob.—*Having two sides and their included angle given in a spherical triangle, to find the area.*

Solution.—Compute the other two angles by Napier's Analogies, and find the area from the angles. [The formula $\cot \frac{1}{2}K = \frac{\cot \frac{1}{2}a \cot \frac{1}{2}b + \cos C}{\sin C}$ gives the spherical excess in terms of two sides and their included angle; but it is of no practical value for finding the area, as it is not adapted to logarithmic computation. For the manner of producing it and several other forms for K, see Todhunter's Spherical Trigonometry, (*103*)].

PRACTICAL APPLICATIONS.

[Note.—The three following problems are given merely to indicate to the student some departments of investigation in which Spherical Trigonometry is of essential service. The two sciences to which this branch of Pure Mathematics is indispensable, are Geodesy, or the mathematical measurement of the earth, and Astronomy.]

Prob. 1.—*To find the shortest distance on the earth's surface between two points whose latitudes and longitudes are known.*

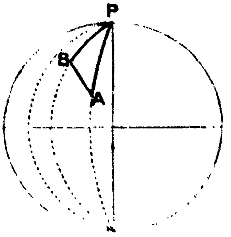


Fig. 71.

Sug's.—The shortest distance on the surface between two points is the arc of a great circle joining the points. Hence, the Problem is: Given two sides (the co-latitudes) and the included angle (the difference in longitude), to find the third side.

Ex. 1. Berlin is situated in Lat. 52° 31′ 13″ N., Lon. 13° 23′ 52″ E., and Alexandria, Egypt, in Lat. 31° 13′ N., Lon. 29° 55′ E. What is the shortest distance in miles on the earth's surface between them, the earth being considered a sphere whose radius is 3962 miles?

Ans. 1692.44 miles.

Therefore ½ the hour angle NNS′ = 30° 22′ 03″.3, and the hour angle is 60° 44′ 07″. This reduced to time at 4 minutes to a degree, gives 4 h. 2 m. 56 s. before noon, or 7 h. 57 m. 4 s. A. M.

Ex. 2. In latitude 40° 21′ N., when the declination of the sun is 3° 20′ S., and its altitude 36° 12′, what is the time of day?

Ans. 9 h. 43 m. 44 s. A. M.

Ex. 3. In latitude 21° 02′ S., when the sun's declination was 18° 32′ N., and the altitude in the afternoon 40° 08′, what was the time of day? *Ans.* 2 h. 2 m. P. M.

***Prob. 3.**—To find the time of sunrising and sunsetting at any given place on a given day.*

SUG'S.—The projection being the same as before, let M′RS′M represent the apparent diurnal path of the sun. Since S′M is described in 6 hours, the time taken to describe RS′ is the time before 6 o'clock, at which the sun rises, *i. e.*, passes the horizon HO. But the time requisite to describe RS′, is the same part of 24 hours (360° angular measure) that the angle CNL (= arc CL) is of 360°. Hence, the arc CL, in time, is the time before 6 o'clock at which the sun rises. In a similar manner, C*l*, in time, is seen to be the time *after* 6 o'clock when the sun is south of the equator. The solution of the problem, therefore, consists in finding CL. Now, in the triangle RLC, right angled at L, LR = the sun's declination at the time, and angle RCL = ECH = the co-latitude of the place.* From these data CL is readily found.

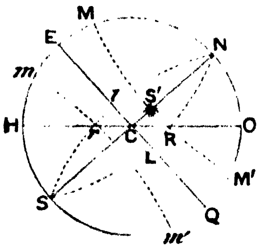


FIG. 73.

Ex. 1. Required the time of sunrise at latitude 42° 33′ N., when the sun's declination is 13° 28′ N.

COMPUTATION.

cot 47° 27′ = 9.962813
tan 13° 28′ = 9.379239
sin 12° 41′ 52″ = $9.34205\bar{2}$

(12° 41′ 52″) × 4 gives the time before 6 o'clock as 50′ 47″. ∴ The sun rises at 5 h. 09 m. 13 s.

---

[*] This may be seen thus: Suppose a person to start from the equator at E and travel north. When he is at E, the south point of his horizon (H) is at S; and for every degree he goes north, the south pole (S) sinks a degree below his horizon. Hence, HCS = his latitude, and ECH = co-latitude.

Ex. 2. Required the time of sunrise at latitude 57° 02′ 54″ N., when the sun's declination is 23° 28′ N.

Sun rises at 3 h. 11 m. 49 s.

Ex. 3. How long is the sun above the horizon in latitude 58° 12′ N., when the sun's declination is 18° 41′ S., that is about January 25th?

Ans. 7 h. 35 m. 36 s.

Ex. 4. What is the length of the longest day at Ann Arbor, Mich., Lat. 42° 16′ 48″.3, the sun's greatest declination being 23° 27′?

Ans. 15 h. 05 m. 50 s.

[NOTE.—In such problems as the foregoing, several small corrections have to be made in order to entire accuracy, such, for example, as that for refraction in taking the altitude, and for the time required for the sun's disk to pass the horizon. But they would be out of place here.]

THE END.